J. Fluid Mech. (1989), *vol.* 209, *pp.* 435–462
Printed in Great Britain

Inertia effects on the motion of long slender bodies

By R. E. KHAYAT† AND R. G. COX
Pulp and Paper Research Institute of Canada *and* Department of Civil Engineering and Applied Mechanics, McGill University, Montreal, Canada H3A 2K6

(Received 5 April 1988 and in revised form 7 April 1989)

A solid long slender body with curved centreline is held at rest in a fluid undergoing a uniform flow. Assuming that the Reynolds number *Re* based on body length is fixed, the force per unit length on the body is obtained as an asymptotic expansion in terms of the ratio κ of the cross-sectional radius to body length. In the limit of large *Re*, this result is no longer valid and an asymptotic expansion in κRe is necessary. A uniformly valid solution is obtained from these two expansions. The total force and torque acting on a body with a straight centreline are explicitly determined. The limiting cases of small and large *Re* are studied in detail.

1. Introduction

The motion of a fluid around a single isolated body has long received considerable attention. While flow at low Reynolds number has been extensively studied analytically (see for example Happel & Brenner 1963), the case of high-Reynolds-number flow has been studied mostly by using numerical methods because of the difficulty in dealing with the nonlinear inertia terms in the Navier–Stokes equation.

However, by making an expansion in terms of the Reynolds number (assumed small) the hydrodynamic force on a sphere and on an infinite cylinder placed in a uniform flow was obtained by Proudman & Pearson (1957). The results for the sphere were extended by Brenner & Cox (1963) to bodies of arbitrary shape.

Since the cases of moderate and high Reynolds number are of practical importance, it is of interest to investigate the flow around a class of bodies of irregular shape for which one may solve analytically the flow equations including inertia effects. In this paper, we calculate the hydrodynamic force acting on a long slender solid body of arbitrary cross-sectional shape held fixed in a uniform flow field. The body centreline need not necessarily be straight. If such a body is of length $2a$ and has a characteristic cross-sectional lengthscale b, expansions of the velocity and pressure fields for the flow about such a body are made in terms of the parameter $\kappa = b/a$. Inertia effects are included with the Reynolds number *Re* based on the body half-length a being arbitrary and not restricted to any particular range of values. However, the Reynolds number R based on the cross-sectional dimension b is assumed small. Such a body is a suitable model for a rigid fibre or thread-like particle. After the detailed statement of the problem considered (§2), the hydrodynamic force per unit length on the particle is calculated (§§3–5) by the method of matched asymptotic expansions. Results for a particle with straight centreline and circular cross-section are then considered in §6. It is then shown that in the case of a body of infinite length, the expansion in terms of κ ceases to be valid. Instead, an expansion in R is necessary. This expansion is obtained in §7 together with a solution which is universally valid

† Present address: Department of Chemistry, McGill University.

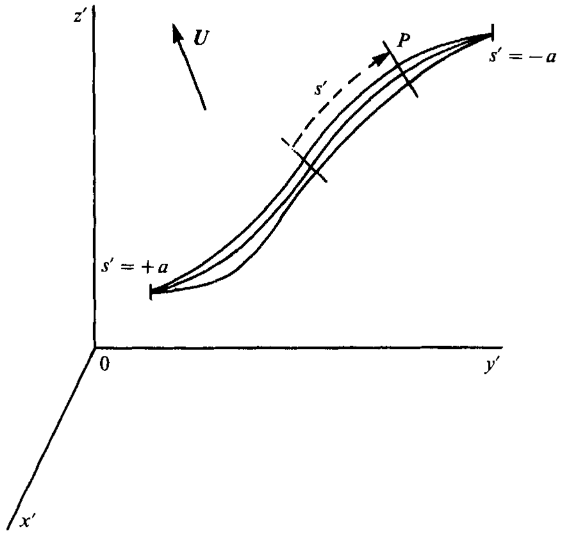

FIGURE 1. A long slender body at rest in a fluid with an undisturbed uniform flow U.

in the double limit $\kappa \to 0$, $R \to 0$. The results for the force per unit length and the total drag, lift and torque on the particle for various limiting cases are discussed in §8. In particular a discrepancy between the present results and those obtained by Chwang & Wu (1976) for a slender spheroidal particle is explained.

2. The general problem

Consider a long slender body whose cross-sectional shape is non-circular and varies along the body centreline. The length of the body is $2a$ and the characteristic dimension of the cross-sectional shape is b. The body centreline may be assumed bent in any manner whatsoever as long as the radius of curvature of such a bending is at all points of order a. The distance along the body centreline measured from the centreline midpoint is s' (see figure 1).

Locally at a general point P on the body centreline we define a set of Cartesian axes $(\bar{x}', \bar{y}', \bar{z}')$ and a set of cylindrical polar coordinates $(\bar{\rho}', \theta, \bar{z}')$ with origin at P and the $\bar{z}'$ axis tangent to the body centreline as shown in figure 2. The cross-sectional shape of the body at P may be written as $\bar{\rho}' = b\lambda(s, \theta)$, where λ is a dimensionless function of s and θ. Here s is the dimensionless distance along the body centreline given by

$$s = s'/a \tag{2.1}$$

so that $-1 \leqslant s \leqslant 1$ with the two ends of the body being $s = -1$ and $s = 1$. We assume that the ratio $\kappa = b/a \ll 1$, i.e. the body is slender.

The body is considered at rest in a fluid (of viscosity μ and density ρ) in which there is a uniform undisturbed flow field of (dimensional) velocity $\boldsymbol{U}$. Associated with $\boldsymbol{U}$ is the constant free-stream pressure which, without loss of generality, may be taken to be zero. We are interested in obtaining the drag force on the body in the limit as $\kappa \to 0$ with the Reynolds number $Re \equiv \rho|\boldsymbol{U}|a/\mu$ based on the body length assumed to be of order unity. The Reynolds number $R \equiv \rho|\boldsymbol{U}|b/\mu \equiv \kappa Re$ based on the body cross-sectional dimension then tends to zero. It is in terms of the parameter κ that we shall make expansions of the velocity and pressure fields. However, one should note that

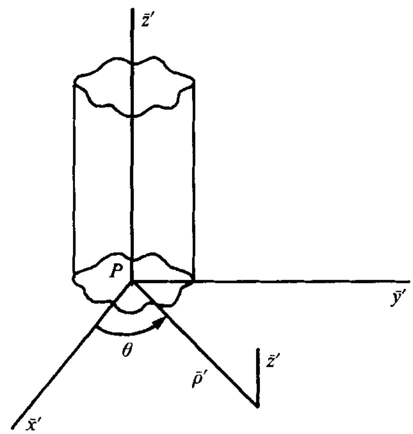

FIGURE 2. The cylindrical coordinate system $(\bar{\rho}', \theta, \bar{z}')$ showing local cross-sectional shape.

this type of expansion must be singular since the flow locally around the long slender body must be very nearly the flow around an infinite cylinder at zero Reynolds number R, and it is well known (Stokes' paradox) that it is impossible for such a flow field to satisfy the equations of motion and at the same time to satisfy the no-slip condition on the surface of the infinite cylinder and also to make the velocity tend to the uniform flow at infinity.

The dimensionless position vector $\boldsymbol{r}$, flow velocity $\boldsymbol{u}$ and pressure p are defined in terms of the corresponding dimensional quantities $\boldsymbol{r}'$, $\boldsymbol{u}'$ and p' as follows:

$$\boldsymbol{r} = \frac{\boldsymbol{r}'}{a}, \quad \boldsymbol{u} = \frac{\boldsymbol{u}'}{U}, \quad p = \frac{ap'}{\mu U}. \tag{2.2}$$

where $U = |\boldsymbol{U}|$. Unless otherwise stated, we use unprimed variables to denote dimensionless quantities. The vector $\boldsymbol{r}$ is the dimensionless position vector of a general point relative to a fixed set of rectangular Cartesian coordinates with origin 0 (see figure 1) so that the body centreline itself is given by $\boldsymbol{r} = \boldsymbol{R}(s)$. The governing equations of momentum and continuity for $(\boldsymbol{u}, p)$, in dimensionless form are

$$Re\, \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla^2\boldsymbol{u} - \nabla p; \quad \nabla\cdot\boldsymbol{u} = 0. \tag{2.3a}$$

We intend to solve (2.3a) as an expansion in κ using the boundary conditions

$$\boldsymbol{u} \to \boldsymbol{e} \quad \text{as} \quad \boldsymbol{r} \to \infty \tag{2.3b}$$

and

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on the body surface,} \tag{2.3c}$$

where $\boldsymbol{e}$ is the unit vector in the direction of the uniform undisturbed flow. This will require obtaining a solution as an outer expansion in κ valid in a region (the outer region), where $\boldsymbol{r}$ is of order unity, so that in this region lengths are made dimensionless by a and, as $\kappa \to 0$, the body becomes a line singularity (see figure 3).

At each point P (at $\boldsymbol{r} = \boldsymbol{R}_P$) of the body centreline one may define an inner expansion in κ for which $\bar{\boldsymbol{r}}$ is used as the independent variable and $\bar{\boldsymbol{u}}$ and $\bar{p}$ as dependent variables, where $\bar{\boldsymbol{r}}$, $\bar{\boldsymbol{u}}$ and $\bar{p}$ are given by

$$\bar{\boldsymbol{r}} = \frac{\boldsymbol{r} - \boldsymbol{R}_P}{\kappa}, \quad \bar{\boldsymbol{u}} = \boldsymbol{u}, \quad \bar{p} = \kappa p. \tag{2.4}$$

We write $\bar{\boldsymbol{r}} = (\bar{x}, \bar{y}, \bar{z})$, where the $\bar{x}$, $\bar{y}$ and $\bar{z}$ are respectively $\bar{x}'$, $\bar{y}'$ and $\bar{z}'$ made dimensionless by b. In the inner expansion corresponding to a point P of the body

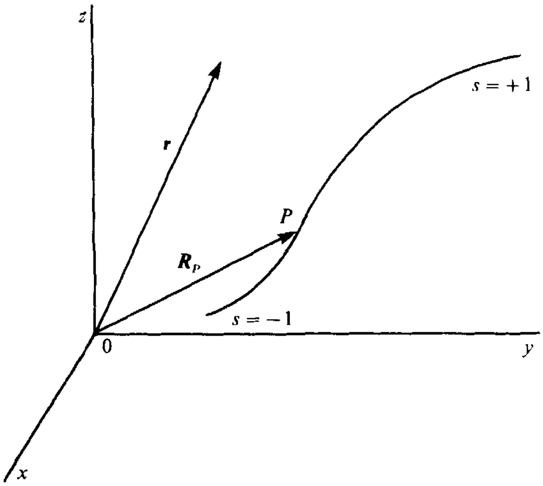

FIGURE 3. The outer region in which lengths are made dimensionless by a. The body becomes a line singularity as $\kappa \to 0$.

centreline, lengths are made dimensionless by b so that, as $\kappa \to 0$, the body becomes very much like a non-circular cylinder of infinite length. Actually, one has an infinite number of inner expansions corresponding to each point of the line singularity representing the body in the outer expansion. However, all such inner expansions may be considered simultaneously by taking a general point P of the line singularity. The inner expansion at such a point is then matched onto the solution of the outer expansion at the same point P.

3. Matched asymptotic expansion

3.1. *Inner expansion*

At a general point P of the body centreline consider now the inner expansion. The flow field $(\bar{\boldsymbol{u}}, \bar{p})$ in the vicinity of the point P is to be computed by solving the governing motion equations (2.3) expressed in inner variables, i.e.

$$\kappa Re\, \bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{\nabla}} \bar{\boldsymbol{u}} = \bar{\nabla}^2 \bar{\boldsymbol{u}} - \bar{\boldsymbol{\nabla}} \bar{p}; \quad \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{u}} = 0, \tag{3.1}$$

with $\bar{\boldsymbol{u}} = 0$ on the body surface. Here $\bar{\boldsymbol{\nabla}}$ is the gradient operator with respect to the $(\bar{x}, \bar{y}, \bar{z})$-coordinates.

No outer boundary condition will be imposed on $\bar{\boldsymbol{u}}$ at this stage since this will be determined by matching.

Relative to the inner dimensionless coordinates $\bar{x}$, $\bar{y}$ and $\bar{z}$, a dimensionless cylindrical polar coordinate system $(\bar{\rho}, \theta, \bar{z})$ is defined with $\bar{\rho} = \bar{\rho}'/b$ so that

$$\bar{x} = \bar{\rho} \cos\theta, \quad \bar{y} = \bar{\rho} \sin\theta. \tag{3.2}$$

We suppose that $\lambda(s, \theta)$ varies slowly with s. The value of $\bar{\boldsymbol{u}}$ may then, at lowest order, be calculated in the same manner as for $Re = 0$ (Cox 1970; Batchelor 1970) and be shown to possess for $\bar{\rho} \to \infty$ the asymptotic form

$$(\bar{u}_0)_{\bar{\rho}} \sim \left[2 \ln\left(\frac{R_s}{\bar{\rho}}\right) + 1\right] [C(\kappa) \cos\theta + D(\kappa) \sin\theta] \\ + 2(K_{xx} \cos\theta + K_{yx} \sin\theta)\, C(\kappa) + 2(K_{xy} \cos\theta + K_{yy} \sin\theta)\, D(\kappa), \tag{3.3a}$$

$$(\bar{u}_0)_\theta \sim \left[2\ln\left(\frac{R_s}{\bar{\rho}}\right)-1\right][D(\kappa)\cos\theta - C(\kappa)\sin\theta]$$
$$+2(K_{yx}\cos\theta - K_{xx}\sin\theta)\,C(\kappa)+2(K_{yy}\cos\theta - K_{xy}\sin\theta)\,D(\kappa), \quad (3.3b)$$

$$(\bar{u}_0)_{\bar{z}} = E(\kappa)\ln\left(\frac{\bar{\rho}}{KR_s}\right)+O\left(\frac{1}{\bar{\rho}}\right), \quad (3.3c)$$

$$\bar{p}_0 \sim 4\bar{\rho}^{-1}[C(\kappa)\cos\theta + D(\kappa)\sin\theta]+F(\kappa). \quad (3.3d)$$

Here K is a constant scalar and K_{ij} a constant symmetric tensor whose values depend only on the local cross-sectional shape (but not on the cross-sectional size) at the point of the body considered (see Batchelor 1970). For a circular cross-section $K = 1$ and $K_{ij} = 0$. It will be seen later that $C(\kappa)$, $D(\kappa)$, $E(\kappa)$ and $F(\kappa)$ must be taken as functions of κ. The changing of (3.3) to outer variables yields the inner boundary condition on the flow field $(\boldsymbol{u}, p)$ of the outer expansion. Using for the outer region polar coordinates (ρ, θ, z) corresponding to the $(\bar{\rho}, \theta, \bar{z})$ coordinates where

$$\rho = \kappa\bar{\rho}, \quad z = \kappa\bar{z}, \quad (3.4)$$

the inner boundary conditions on the outer flow field $(\boldsymbol{u}, p)$ may be obtained. In this procedure in order that no terms singular in κ should appear in the outer expansion, it is seen that $C(\kappa)$, $D(\kappa)$, $E(\kappa)$ and $F(\kappa)$ must be of the form (see Cox 1970)

$$C(\kappa) = \frac{C_1}{\ln\kappa}+\frac{C_2}{(\ln\kappa)^2}+\ldots, \quad D(\kappa) = \frac{D_1}{\ln\kappa}+\frac{D_2}{(\ln\kappa)^2}+\ldots, \quad (3.5a, b)$$

$$E(\kappa) = \frac{E_1}{\ln\kappa}+\frac{E_2}{(\ln\kappa)^2}+\ldots, \quad F(\kappa) = \kappa F_0+\frac{\kappa F_1}{\ln\kappa}+\ldots. \quad (3.5c, d)$$

Thus upon the use of (3.3), (3.4) and (3.5), it is seen that in the limit $\rho \to 0$, (i.e. for a point $\boldsymbol{r}$ which moves in towards the singularity line $\boldsymbol{r} = \boldsymbol{R}(s)$ at the point P), the outer flow field $(\boldsymbol{u}, p)$ has the following form:

$$u_\rho \sim 2(C_1\cos\theta + D_1\sin\theta)+\left(\frac{1}{\ln\kappa}\right)\left\{\left[2C_2+C_1-2C_1\ln\left(\frac{\rho}{R_s}\right)\right.\right.$$
$$\left.+2(K_{xx}C_1+K_{xy}D_1)\right]\cos\theta+\left[2D_2+D_1-2D_1\ln\left(\frac{\rho}{R_s}\right)\right.$$
$$\left.\left.+2(K_{yx}C_1+K_{yy}D_1)\right]\sin\theta\right\}+O\left(\frac{1}{\ln\kappa}\right)^2, \quad (3.6a)$$

$$u_\theta \sim -2(C_1\sin\theta - D_1\cos\theta)+\left(\frac{1}{\ln\kappa}\right)\left\{\left[-2C_2+C_1+2C_1\ln\left(\frac{\rho}{R_s}\right)\right.\right.$$
$$\left.-2(K_{xx}C_1-K_{xy}D_1)\right]\sin\theta+\left[2D_2-D_1-2D_1\ln\left(\frac{\rho}{R_s}\right)\right.$$
$$\left.\left.+2(K_{yx}C_1+K_{yy}D_1)\right]\cos\theta\right\}+O\left(\frac{1}{\ln\kappa}\right)^2, \quad (3.6b)$$

$$u_z \sim -E_1+\left(\frac{1}{\ln\kappa}\right)\left[-E_2+E_1\ln\left(\frac{\rho}{KR_s}\right)\right]+O\left(\frac{1}{\ln\kappa}\right)^2, \quad (3.6c)$$

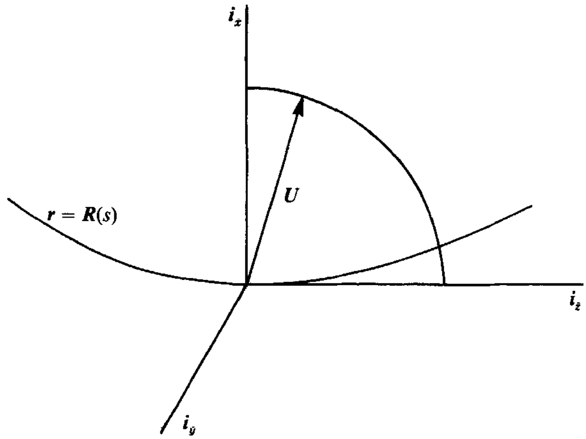

FIGURE 4. The system of axes with unit base vectors $\boldsymbol{i}_{\bar{x}}, \boldsymbol{i}_{\bar{y}}, \boldsymbol{i}_{\bar{z}}$.

and

$$p \sim F_0 + \left(\frac{1}{\ln \kappa}\right)[4\rho^{-1}(C_1 \cos\theta + D_1 \sin\theta) + F_1] + O\left(\frac{1}{\ln \kappa}\right)^2. \tag{3.6d}$$

3.2. *Outer expansion*

We recall that the outer flow field $(\boldsymbol{u}, p)$ satisfies (2.3a) with the outer boundary condition that $\boldsymbol{u} \to \boldsymbol{e}$ as $\boldsymbol{r} \to \infty$ (where $\boldsymbol{e}$ is a unit vector in the undisturbed flow direction). From the form of the inner boundary conditions (3.6) on $(\boldsymbol{u}, p)$ it would seem reasonable (Cox 1970) to assume $(\boldsymbol{u}, \rho)$ has an expansion of the form

$$\boldsymbol{u} = \boldsymbol{e} + \frac{\boldsymbol{u}_1}{\ln \kappa} + O\left(\frac{1}{\ln \kappa}\right)^2, \quad p = \frac{p_1}{\ln \kappa} + O\left(\frac{1}{\ln \kappa}\right)^2, \tag{3.7a, b}$$

where the zeroth-order term represents the free-stream flow. At a general point P on the line singularity $\boldsymbol{r} = \boldsymbol{R}(s)$ it is convenient to take a set of rectangular Cartesian axes with unit base vectors $\boldsymbol{i}_{\bar{z}}$, $\boldsymbol{i}_{\bar{x}}$ and $\boldsymbol{i}_{\bar{y}}$ which lie in the same direction as the $(\bar{z}, \bar{x}, \bar{y})$-axes at P (see §2). Thus $\boldsymbol{i}_{\bar{z}}$ lies in the direction of the tangent to $\boldsymbol{r} = \boldsymbol{R}(s)$ at P. Since the $\bar{x}$ and $\bar{y}$ axes were arbitrary, one may now, for convenience, take $\boldsymbol{i}_{\bar{x}}$ to lie in the plane containing $\boldsymbol{i}_{\bar{z}}$ and the velocity vector $\boldsymbol{e}$ (see figure 4). Thus the unit vectors $\boldsymbol{i}_{\bar{z}}$, $\boldsymbol{i}_{\bar{x}}$ and $\boldsymbol{i}_{\bar{y}}$ are

$$\boldsymbol{i}_{\bar{z}} = \boldsymbol{t}, \quad \boldsymbol{i}_{\bar{x}} = \frac{\boldsymbol{e}\cdot[\boldsymbol{I} - \boldsymbol{tt}]}{[1 - |\boldsymbol{e}\cdot\boldsymbol{t}|^2]^{\frac{1}{2}}}, \quad \boldsymbol{i}_{\bar{y}} = \frac{\boldsymbol{t}\times\boldsymbol{e}}{[1 - |\boldsymbol{e}\cdot\boldsymbol{t}|^2]^{\frac{1}{2}}}, \tag{3.8a, b, c}$$

where $\boldsymbol{t}(s) = \mathrm{d}\boldsymbol{R}/\mathrm{d}s$ is a unit vector in the tangent direction and $\boldsymbol{I}$ is the idemfactor.

The terms in (3.7) of order unity must behave near the line $\boldsymbol{r} = \boldsymbol{R}(s)$ like the terms of order unity in (3.6). Hence

$$e_{\bar{x}} = 2C_1, \quad e_{\bar{y}} = 2D_1, \quad e_{\bar{z}} = -E_1, \quad 0 = F_0, \tag{3.9}$$

where $e_{\bar{x}}$, $e_{\bar{y}}$ and $e_{\bar{z}}$ are the components of $\boldsymbol{e}$ along the $\bar{x}$-, $\bar{y}$- and $\bar{z}$-axes. Therefore, upon using (3.8) the values of C_1, D_1, E_1 and F_0 are given by

$$C_1 = \tfrac{1}{2}[1 - |\boldsymbol{e}\cdot\boldsymbol{t}|^2]^{\frac{1}{2}}, \quad D_1 = 0, \quad E_1 = -\boldsymbol{e}\cdot\boldsymbol{t}, \quad F_0 = 0. \tag{3.10}$$

By matching of terms of order $1/\ln\kappa$ in (3.7), it is seen that the flow field $(\boldsymbol{u}_1, p_1)$ near the line $\boldsymbol{r} = \boldsymbol{R}(s)$ must behave as $\rho \to 0$ like

$$\left.\begin{aligned}(u_1)_\rho &\sim -2C_1\cos\theta\ln\rho + (2C_2 + C_1 + 2C_1\ln R_s(s)\\ &\qquad + 2K_{xx}C_1)\cos\theta + 2(D_2 + K_{yx}C_1)\sin\theta + \dots,\\ (u_1)_\theta &\sim 2C_1\sin\theta\ln\rho + (-2C_2 + C_1 - 2C_1\ln R_s(s)\\ &\qquad - 2K_{xx}C_1)\sin\theta + 2(D_2 + K_{yx}C_1)\cos\theta + \dots,\\ (u_1)_z &\sim E_1\ln\rho - E_2 - E_1\ln KR_s(s) + \dots,\\ p_1 &\sim 4\rho^{-1}C_1\cos\theta + F_1 + \dots .\end{aligned}\right\} \tag{3.11}$$

It can be shown (see Cox 1970) that the singular part of $(\boldsymbol{u}_1, p_1)$ given by (3.11) (i.e. the terms of order $\ln\rho$ for $\boldsymbol{u}_1$ and of order ρ^{-1} for p_1) represents a line of force on $\boldsymbol{r} = \boldsymbol{R}(s)$ with strength $\boldsymbol{f}^*(s)$ per unit length given by

$$\boldsymbol{f}^*(s) = (8\pi C_1)\,\boldsymbol{i}_{\bar{x}} - (2\pi E_1)\,\boldsymbol{i}_{\bar{z}}, \tag{3.12}$$

which by (3.8) and (3.10) becomes

$$\boldsymbol{f}^*(s) = 4\pi[\boldsymbol{e}\cdot(\boldsymbol{I} - \tfrac{1}{2}\boldsymbol{tt})]. \tag{3.13}$$

Upon substituting the expansions (3.7) for $(\boldsymbol{u}, p)$ into (2.3) it is seen that it is identically satisfied at $O(1/\ln\kappa)^0$ but that at $O(1/\ln\kappa)$ one obtains

$$Re\,\boldsymbol{e}\cdot\nabla\boldsymbol{u}_1 = \nabla^2\boldsymbol{u}_1 - \nabla p_1; \quad \nabla\cdot\boldsymbol{u}_1 = 0. \tag{3.14}$$

4. The outer flow solution

The governing equations (3.14) for the outer flow field $(\boldsymbol{u}_1, p_1)$ are to be solved subject to the boundary condition

$$\boldsymbol{u}_1 \to 0 \quad \text{as} \quad \boldsymbol{r} \to \infty \tag{4.1}$$

and representing the line of force $\boldsymbol{f}^*(s)$ on $\boldsymbol{r} = \boldsymbol{R}(s)$. Equation (3.14) is Oseen's equation for a uniform flow in the direction $\boldsymbol{e}$. It possesses a solution for $\boldsymbol{u}_1$ and p_1 for a point force (f_1, f_2, f_3) located at the origin, given by (see Happel & Brenner 1970)†

$$\left.\begin{aligned}(u_1)_i(\boldsymbol{r}) &= \frac{1}{8\pi}\left[f_j\delta_{ij}\nabla^2\psi(\boldsymbol{r}) - f_j\frac{\partial^2\psi(\boldsymbol{r})}{\partial r_i\,\partial r_j}\right],\\ p_1(\boldsymbol{r}) &= \frac{1}{4\pi}\frac{r_j f_j}{r^3},\end{aligned}\right\} \tag{4.2}$$

where r is the radial distance from the origin defined by

$$r = (r_j r_j)^{\frac{1}{2}} = |\boldsymbol{r}|, \tag{4.3}$$

and where $\psi(r)$ is defined by

$$\psi(\boldsymbol{r}) = \frac{2}{Re}\int_0^{\frac{1}{2}Re(r - \boldsymbol{e}\cdot\boldsymbol{r})} \frac{1 - e^{-\alpha}}{\alpha}\,d\alpha. \tag{4.4}$$

† Repeated indices refer to summation over the index unless otherwise specified.

Thus the flow field $(\boldsymbol{u}_1, p_1)$ due to the line of force $\boldsymbol{f}^*(s)$ on $\boldsymbol{r} = \boldsymbol{R}(s)$ is given by

$$\left.\begin{aligned} (u_1)_i &= \frac{1}{8\pi}\int_{-1}^{1} g_{ij}(\boldsymbol{r}-\hat{\boldsymbol{R}}) f_j^*(\hat{s})\,\mathrm{d}\hat{s}, \\ p_1 &= \frac{1}{4\pi}\int_{-1}^{1} \frac{(r_j-\hat{R}_j)}{|\boldsymbol{r}-\hat{\boldsymbol{R}}|^3} f_j^*(\hat{s})\,\mathrm{d}\hat{s}, \end{aligned}\right\} \tag{4.5a}$$

where the function $g_{ij}(\boldsymbol{r})$ is defined by

$$g_{ij}(\boldsymbol{r}) = \delta_{ij}\nabla^2\psi(\boldsymbol{r}) - \frac{\partial^2\psi(\boldsymbol{r})}{\partial r_i \partial r_j}. \tag{4.5b}$$

The integration is taken over the line $-1 \leqslant \hat{s} \leqslant 1$ and $\hat{\boldsymbol{R}}$ represents the value of $\boldsymbol{r}$ at a point $s = \hat{s}$ on the line of force. The substitution of the value of $\boldsymbol{f}^*(s)$ from (3.13) into the above expressions yields

$$\left.\begin{aligned} (u_1)_i &= \frac{1}{2}\int_{-1}^{1} g_{ij}(\boldsymbol{r}-\hat{\boldsymbol{R}})\,(\delta_{jk}-\tfrac{1}{2}t_j(\hat{s})\,t_k(\hat{s}))\,e_k\,\mathrm{d}\hat{s}, \\ p_1 &= \int_{-1}^{1} \frac{(r_j-\hat{R}_j)}{|\boldsymbol{r}-\hat{\boldsymbol{R}}|^3}(\delta_{jk}-\tfrac{1}{2}t_j(\hat{s})\,t_k(\hat{s}))\,e_k\,\mathrm{d}\hat{s}. \end{aligned}\right\} \tag{4.6}$$

In order to match this value $(\boldsymbol{u}_1, p_1)$ onto that of the inner expansion, one requires the asymptotic behaviour of $\boldsymbol{u}_1$, p_1) near the line singularity $\boldsymbol{r} = \boldsymbol{R}(s)$. Since the integrands for $\boldsymbol{u}_1$ and p_1 become singular on the line of force, we write

$$(u_1)_i = J_i + J_i^*, \quad p_1 = H + H^*, \tag{4.7}$$

where J_i and H are the integrals taken over the intervals $(-1, s-\epsilon)$ and $(s+\epsilon, 1)$ whilst J_i^* and H^* are the integrals taken over the remaining interval $(s-\epsilon, s+\epsilon)$. The quantity ϵ is assumed to be an arbitrary constant (independent of κ) very much smaller than unity. Since the integrands only become singular at $\hat{s} = s$ if $\boldsymbol{r}$ lies on the line singularity, it follows that the integrals J_i and H have integrands with no singularity, although the values of these integrals will tend to infinity as $\epsilon \rightarrow 0$. Since $\epsilon \ll 1$, the integrals J_i^* and H^* may be simplified if one notes that $s \approx \hat{s}$ in the range of integration. Therefore,

$$J_i^* = \tfrac{1}{2}(\delta_{jk} - \tfrac{1}{2}t_j t_k)\,e_k I_{ij}, \quad H^* = (\delta_{jk} - \tfrac{1}{2}t_j t_k)\,e_k I_j, \tag{4.8}$$

where

$$I_{ij} = \int_{s-\epsilon}^{s+\epsilon} g_{ij}(\boldsymbol{r}-\hat{\boldsymbol{R}})\,\mathrm{d}\hat{s}, \quad I_i = \int_{s-\epsilon}^{s+\epsilon} \frac{r_i - \hat{R}_i}{|r-\hat{R}|^3}\,\mathrm{d}\hat{s}. \tag{4.9}$$

For fixed but small ϵ, as we approach the line singularity, i.e. in the limit as $\rho \rightarrow 0$, the flow approaches that due to a line of constant force acting on the z-axis. Thus one may obtain asymptotic forms of I_{ij} and I_i for $\rho \rightarrow 0$. Then from (4.7) and (4.8) the asymptotic forms of $\boldsymbol{u}_1$ and p_1 may be obtained as

$$\left.\begin{aligned} (u_1)_\rho &\sim -e_x\cos\theta\ln\rho + e_x(\ln 2\epsilon+1)\cos\theta + J_x\cos\theta + J_y\sin\theta, \\ (u_1)_\theta &\sim e_x\sin\theta\ln\rho - e_x\ln 2\epsilon\sin\theta - J_x\sin\theta + J_y\cos\theta, \\ (u_1)_z &\sim e_z(-\ln\rho + \ln 2\epsilon - \tfrac{1}{2}) + J_z, \\ p_1 &\sim 2e_x\rho^{-1}\cos\theta + H \end{aligned}\right\} \tag{4.10}$$

for $\rho \rightarrow 0$.

Comparing these equations with the asymptotic form (3.11) of $(\boldsymbol{u}_1, p_1)$ near $\boldsymbol{r} = \boldsymbol{R}(s)$ obtained from the outer limit of the inner solutions, it is observed that the terms of order $\ln \rho$ in $\boldsymbol{u}_1$ and order ρ^{-1} in p_1 are identical (as they must be since solutions have already been matched to this order). We obtain, on matching terms of order ρ^0,

$$\left.\begin{aligned} e_x(\ln 2\epsilon + 1) + J_x &= 2C_2 + C_1 + 2C_1 \ln R_s + 2(K_{xx} C_1 + K_{xy} D_1), \\ J_y &= 2D_2 + 2K_{yx} C_1, \\ e_z(\ln 2\epsilon - \tfrac{1}{2}) + J_z &= -E_2 - E_1 \ln (KR_s), \\ H &= F_1. \end{aligned}\right\} \tag{4.11}$$

By making use of the expressions (3.9) for C_1, D_1 and E_1, the values of C_2, D_2 and E_2 may be obtained from (4.11). These, when substituted into (3.5), give the following expressions:

$$\left.\begin{aligned} C(\kappa) &= \tfrac{1}{2} e_x (\ln \kappa)^{-1} + \frac{1}{2}\left[e_x\left(\ln \frac{2\epsilon}{R_s} + \tfrac{1}{2} - K_{xx}\right) + J_x\right](\ln \kappa)^{-2} + \dots, \\ D(\kappa) &= \tfrac{1}{2}(J_y - e_x K_{yx})(\ln \kappa)^{-2} + \dots, \\ E(\kappa) &= -e_z(\ln \kappa)^{-1} + \left[e_z\left(\ln \frac{KR_s}{2\epsilon} + \frac{1}{2}\right) - J_z\right](\ln \kappa)^{-2} + \dots, \\ F(\kappa) &= \kappa K (\ln \kappa)^{-1} + \dots. \end{aligned}\right\} \tag{4.12}$$

5. Force on body

The total force per unit length acting on the body can be found from the asymptotic form of the inner flow field, as $\bar{\rho} \to \infty$, or equivalently the outer flow, as $\rho \to 0$. The evaluation of the force is carried out in a manner similar to Cox (1970), where it is shown that the dimensional force per unit length $\boldsymbol{f}(s)$ is given by

$$\frac{\boldsymbol{f}(s)}{2\pi\mu U} = -4C(\kappa)\,\boldsymbol{i}_{\bar{x}} - 4D(\kappa)\,\boldsymbol{i}_{\bar{y}} + E(\kappa)\,\boldsymbol{i}_{\bar{z}}. \tag{5.1}$$

If we substitute the expressions (4.12) for $C(\kappa)$, $D(\kappa)$ and $E(\kappa)$ into (5.1), and make use of (3.8), we obtain $\boldsymbol{f}(s)$ as

$$\begin{aligned} \frac{\boldsymbol{f}(s)}{2\pi\mu U} = \boldsymbol{e}\cdot[\boldsymbol{tt} - 2\boldsymbol{I}]\left(\frac{1}{\ln \kappa}\right) + \Bigg[&\left(\boldsymbol{J} + \boldsymbol{e} \ln \frac{2\epsilon}{R_s}\right)\cdot(\boldsymbol{tt} - 2\boldsymbol{I}) \\ &+ \boldsymbol{e}\cdot(\tfrac{3}{2}\boldsymbol{tt} - \boldsymbol{I}) + 2\boldsymbol{e}\cdot\boldsymbol{K} + \boldsymbol{e}\cdot\boldsymbol{tt}\ln K\Bigg]\left(\frac{1}{\ln \kappa}\right)^2 + O\left(\frac{1}{\ln \kappa}\right)^3, \end{aligned} \tag{5.2}$$

where $\boldsymbol{J}$ is by definition a vector given by

$$J_i = \frac{1}{2}\left(\int_{-1}^{s-\epsilon} + \int_{s+\epsilon}^{1}\right) g_{ij}(\boldsymbol{R} - \hat{\boldsymbol{R}})\,(\delta_{jk} - \tfrac{1}{2} t_j(\hat{s})\, t_k(\hat{s}))\, e_k \,\mathrm{d}\hat{s}, \tag{5.3}$$

where $\boldsymbol{R}$ is the value of $\boldsymbol{r}$ at the point on the centreline under consideration and $\hat{\boldsymbol{R}}$ is the value at the point on the centreline with $s = \hat{s}$. Substitution of the value of g_{ij}, obtained from (4.4) and (4.5b) yields the value of J_i as

$$J_i(\boldsymbol{R}) = \frac{1}{2}\left(\int_{-1}^{s-\epsilon} + \int_{s+\epsilon}^{1}\right)\left\{\frac{2[1-\mathrm{e}^{-\frac{1}{2}Re\,[|\boldsymbol{R}-\hat{\boldsymbol{R}}|-\boldsymbol{e}\cdot(\boldsymbol{R}-\hat{\boldsymbol{R}})]}]}{Re\,|\boldsymbol{R}-\hat{\boldsymbol{R}}|^2\,[|\boldsymbol{R}-\hat{\boldsymbol{R}}|-\boldsymbol{e}\cdot(\boldsymbol{R}-\hat{\boldsymbol{R}})]}\left(2\delta_{ij}|\boldsymbol{R}-\hat{\boldsymbol{R}}| - \frac{1}{|\boldsymbol{R}-\hat{\boldsymbol{R}}|}[\delta_{ij}|\boldsymbol{R}-\hat{\boldsymbol{R}}|^2-(R_i-\hat{R}_i)(R_j-\hat{R}_j)] - \frac{1}{|\boldsymbol{R}-\hat{\boldsymbol{R}}|-\boldsymbol{e}\cdot(\boldsymbol{R}-\hat{\boldsymbol{R}})}[\delta_{ij}|\boldsymbol{R}-\hat{\boldsymbol{R}}-|\boldsymbol{R}-\hat{\boldsymbol{R}}|\,\boldsymbol{e}|^2 - (R_i-\hat{R}_i-|\boldsymbol{R}-\hat{\boldsymbol{R}}|\,e_i)(R_j-\hat{R}_j-|\boldsymbol{R}-\hat{\boldsymbol{R}}|\,e_j)]\right) + \frac{\mathrm{e}^{-\frac{1}{2}Re\,[|\boldsymbol{R}-\hat{\boldsymbol{R}}|-\boldsymbol{e}\cdot(\boldsymbol{R}-\hat{\boldsymbol{R}})]}}{|\boldsymbol{R}-\hat{\boldsymbol{R}}|^2\,[|\boldsymbol{R}-\hat{\boldsymbol{R}}|-\boldsymbol{e}\cdot(\boldsymbol{R}-\hat{\boldsymbol{R}})]}[\delta_{ij}|\boldsymbol{R}-\hat{\boldsymbol{R}}-|\boldsymbol{R}-\hat{\boldsymbol{R}}|\,\boldsymbol{e}|^2 - (R_i-\hat{R}_i-|\boldsymbol{R}-\hat{\boldsymbol{R}}|\,e_i)(R_j-\hat{R}_j-|\boldsymbol{R}-\hat{\boldsymbol{R}}|\,e_j)]\right\}(e_j-\tfrac{1}{2}t_j(\hat{s})\,t_k(\hat{s})\,e_k)\,\mathrm{d}\hat{s}. \quad (5.4)$$

It should be pointed out that the value of $\boldsymbol{f}(s)$ given by (5.2) into which the above value of $\boldsymbol{J}$ is substituted must be independent of the arbitrary small parameter ϵ with the term $(\ln\epsilon)\,\boldsymbol{e}\cdot(\boldsymbol{tt}-2\boldsymbol{I})$ in (5.2) cancelling with the term in $(\ln\epsilon)$ in the asymptotic form of $\boldsymbol{J}$ for $\epsilon\rightarrow 0$.

6. Straight centreline with arbitrary orientation

The results given in the previous section for the force acting on a long slender body will now be used to determine the drag force for translation in an arbitrary direction of a long slender body for which the body centreline is straight. The cross-section is considered to be of arbitrary shape. We define a set of rectangular Cartesian axes (r_1, r_2, r_3) with origin 0 at the midpoint of the centreline and with the r_1-axis in the flow direction. The body centreline, given by $\boldsymbol{r} = \boldsymbol{R}(s)$, is thus

$$R_i(s) = s\beta_i, \quad (6.1)$$

where $\boldsymbol{\beta}$ is the unit vector along the body axis (see figure 5). The dimensionless resistance force per unit length $\boldsymbol{f}(s)$ given by (5.2) and (5.4) may, after a considerable amount of calculation, be shown to be

$$\frac{\boldsymbol{f}(s)}{2\pi\mu U} = \left(\frac{1}{\ln\kappa}\right)(\cos\theta\boldsymbol{\beta}-2\boldsymbol{e}) + \left(\frac{1}{\ln\kappa}\right)^2\left\{\tfrac{1}{4}[2\cos\theta\boldsymbol{e}-(2-\cos\theta+\cos^2\theta)\boldsymbol{\beta}] \times\left[\frac{1-\mathrm{e}^{-\frac{1}{2}Re\,(1-\cos\theta)\,(1+s)}}{\frac{1}{2}Re\,(1-\cos\theta)\,(1+s)}-1\right] - \tfrac{1}{4}[2\cos\theta\boldsymbol{e}-(2+\cos\theta+\cos^2\theta)\boldsymbol{\beta}] \times\left[\frac{1-\mathrm{e}^{-\frac{1}{2}Re\,(1+\cos\theta)\,(1-s)}}{\frac{1}{2}Re\,(1+\cos\theta)\,(1-s)}-1\right] - \tfrac{1}{2}(\cos\theta\boldsymbol{\beta}-2\boldsymbol{e})\,[E_1[\tfrac{1}{2}Re\,(1-\cos\theta)\,(1+s)] + \ln(1-\cos\theta)] - \tfrac{1}{2}(\cos\theta\boldsymbol{\beta}-2\boldsymbol{e})\,[E_1[\tfrac{1}{2}Re\,(1+\cos\theta)\,(1-s)] + \ln(1+\cos\theta)] - (\cos\theta\boldsymbol{\beta}-2\boldsymbol{e})\,(\gamma+\ln\tfrac{1}{4}Re\,R_s) + \tfrac{3}{2}\cos\theta\boldsymbol{\beta} - \boldsymbol{e} + 2\boldsymbol{e}\cdot\boldsymbol{K} + \cos\theta\boldsymbol{\beta}\ln K\right\} + O\left(\frac{1}{\ln\kappa}\right)^3, \quad (6.2)$$

where γ is Euler's constant,

$$E_1(x) = \int_x^\infty \frac{\mathrm{e}^{-\tau}}{\tau}\,\mathrm{d}\tau$$

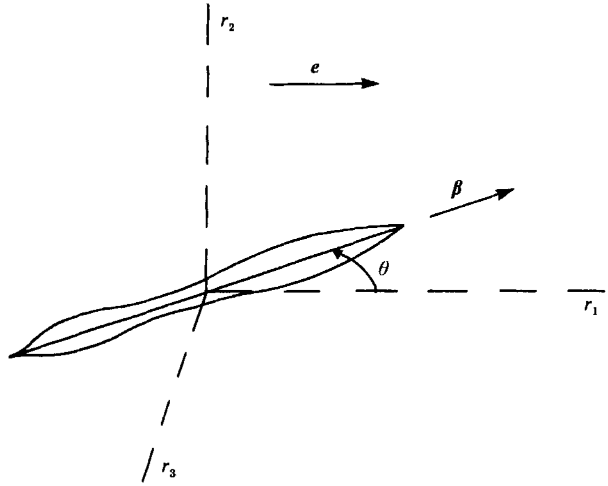

FIGURE 5. The orientation of the body with respect to the free stream.

is the exponential integral and where θ is the angle between the free-stream direction $\boldsymbol{e}$ and the body axis $\boldsymbol{\beta}$, i.e. $\boldsymbol{e}\cdot\boldsymbol{\beta} = \cos\theta$. The total dimensional force $\boldsymbol{F}$ acting on the body

$$\boldsymbol{F} = a\int_{-1}^{1} \boldsymbol{f}(s)\,\mathrm{d}s \tag{6.3}$$

is therefore

$$\begin{aligned}\frac{\boldsymbol{F}}{2\pi\mu Ua} = &\left(\frac{2}{\ln\kappa}\right)(\cos\theta\boldsymbol{\beta}-2\boldsymbol{e})+\left(\frac{1}{\ln\kappa}\right)^2\left\{\frac{2\cos\theta\boldsymbol{e}-(2-\cos\theta+\cos^2\theta)\boldsymbol{\beta}}{2Re\,(1-\cos\theta)}[E_1[Re\,(1-\cos\theta)]\right.\\
&+\ln[Re\,(1-\cos\theta)]+\gamma-Re\,(1-\cos\theta)]-\frac{2\cos\theta\boldsymbol{e}-(2+\cos\theta+\cos^2\theta)\boldsymbol{\beta}}{2Re\,(1+\cos\theta)}\\
&\times[E_1[Re\,(1+\cos\theta)]+\ln[Re\,(1+\cos\theta)]+\gamma-Re\,(1+\cos\theta)]\\
&-(\cos\theta\boldsymbol{\beta}-2\boldsymbol{e})\,[E_1[Re\,(1-\cos\theta)]+\ln(1-\cos\theta)+E_1[Re\,(1+\cos\theta)]\\
&+\ln(1+\cos\theta)+\frac{1-\mathrm{e}^{-Re\,(1-\cos\theta)}}{Re\,(1-\cos\theta)}+\frac{1-\mathrm{e}^{-Re\,(1+\cos\theta)}}{Re\,(1+\cos\theta)}+\int_{-1}^{1}\ln R_s\,\mathrm{d}s\\
&+2(\gamma+\ln\tfrac{1}{4}Re)]+3\cos\theta\boldsymbol{\beta}-2\boldsymbol{e}+2\boldsymbol{e}\cdot\int_{-1}^{1}\boldsymbol{K}(s)\,\mathrm{d}s\\
&\left.+\cos\theta\boldsymbol{\beta}\int_{-1}^{1}\ln K(s)\,\mathrm{d}s\right\}+O\left(\frac{1}{\ln\kappa}\right)^3.\end{aligned} \tag{6.4}$$

The torque $\boldsymbol{G}$ acting on the body about the origin (the midpoint of the body centreline) is

$$G_i = e_{ijk}\,\beta_j\,a^2\int_{-1}^{1} sf_k(s)\,\mathrm{d}s, \tag{6.5}$$

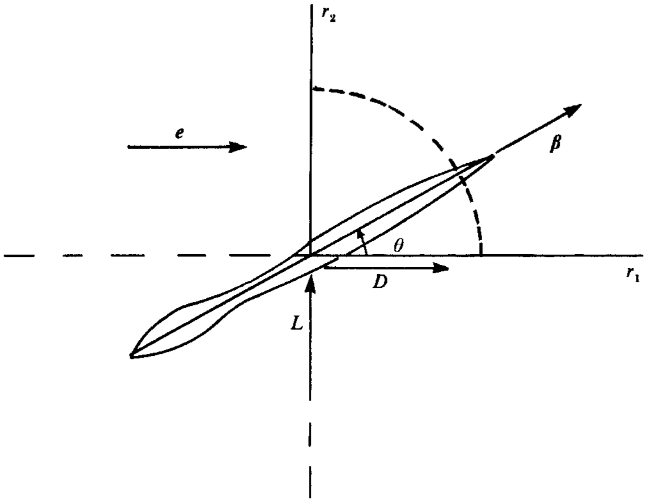

FIGURE 6. The drag and lift components acting on the body.

where e_{ijk} is the permutation symbol. Thus, upon the use of the expression for $f(s)$ from (6.2), one obtains the expressions for the torque as

$$\frac{\boldsymbol{G}}{2\pi\mu Ua^2} = \left(\frac{1}{\ln\kappa}\right)^2\left(\left\{\frac{\cos\theta}{Re\,(1-\cos\theta)}\right.\right.$$
$$\times\left[2+2\frac{\mathrm{e}^{-Re\,(1-\cos\theta)}-1}{Re\,(1-\cos\theta)}-E_1[Re\,(1-\cos\theta)]-\ln[Re\,(1-\cos\theta)]-\gamma\right]$$
$$+\frac{\cos\theta}{Re\,(1+\cos\theta)}\left[2+2\frac{\mathrm{e}^{-Re\,(1+\cos\theta)}-1}{Re\,(1+\cos\theta)}-E_1[Re\,(1+\cos\theta)]\right.$$
$$\left.-\ln[Re\,(1+\cos\theta)]-\gamma\right]$$
$$+2\left[-\frac{1}{Re\,(1-\cos\theta)}\left(1-\frac{1}{Re\,(1-\cos\theta)}+\frac{\mathrm{e}^{-Re\,(1-\cos\theta)}}{Re\,(1-\cos\theta)}\right)\right.$$
$$+\frac{1}{Re\,(1+\cos\theta)}\left(1-\frac{1}{Re\,(1+\cos\theta)}+\frac{\mathrm{e}^{-Re\,(1+\cos\theta)}}{Re\,(1+\cos\theta)}\right)$$
$$\left.\left.\left.+\int_{-1}^{1}s\ln R_s(s)\,\mathrm{d}s\right]\right\}\boldsymbol{\beta}\times\boldsymbol{e}+2\boldsymbol{\beta}\times\left[\boldsymbol{e}\cdot\int_{-1}^{1}s\boldsymbol{K}(s)\,\mathrm{d}s\right]\right)+O\left(\frac{1}{\ln\kappa}\right)^3. \quad (6.6)$$

6.1. *The special case of circular cross-section*

For a body with circular cross-sectional shape we have

$$\boldsymbol{K}(s)=0 \quad \text{and} \quad \ln K(s)=0 \quad \text{for} \quad -1\leqslant s\leqslant 1 \quad (6.7)$$

with $R_s(s)$ now being the cross-sectional radius at position s. If the body centreline is taken to lie in the (r_1, r_2)-plane, the total force $\boldsymbol{F}$ acting on the body can be decomposed into a drag component D parallel to the flow and lift component L perpendicular to the flow (in the r_2-direction) (see figure 6). Expressions for D and L are thus obtained from (6.4) as

$$\frac{D}{\mu Ua} = \frac{-4\pi(2-\cos^2\theta)}{\ln\kappa+F_D(Re\,;\theta)+\dfrac{1}{2}\displaystyle\int_{-1}^{1}\ln R_s(s)\,\mathrm{d}s}+O\left(\frac{1}{\ln\kappa}\right)^3, \quad (6.8)$$

where

$$F_D(Re;\theta) = \Bigg\{\frac{\cos^2\theta}{2Re}[E_1(Re(1-\cos\theta)]+\ln[Re(1-\cos\theta)]+\gamma-Re(1-\cos\theta)]$$
$$+\frac{\cos^2\theta}{2Re}[E_1[Re(1+\cos\theta)]+\ln[Re(1+\cos\theta)]+\gamma-Re(1+\cos\theta)]$$
$$+(2-\cos^2\theta)[E_1[Re(1-\cos\theta)]-\frac{e^{-Re(1-\cos\theta)}-1}{Re(1-\cos\theta)}+\ln(1-\cos\theta)$$
$$+E_1[Re(1+\cos\theta)]-\frac{e^{-Re(1+\cos\theta)}-1}{Re(1+\cos\theta)}+\ln(1+\cos\theta)$$
$$+2(\gamma+\ln\tfrac{1}{4}Re)]+3\cos^2\theta-2\Bigg\}\frac{1}{2(2-\cos^2\theta)}, \tag{6.9}$$

and

$$\frac{L}{\mu Ua} = \frac{+2\pi\sin 2\theta}{\ln\kappa+F_L(Re;\theta)+\frac{1}{2}\int_{-1}^{1}\ln R_s(s)\,\mathrm{d}s}+O\left(\frac{1}{\ln\kappa}\right)^3, \tag{6.10}$$

where

$$F_L(Re;\theta) = \Bigg\{\frac{(2-\cos\theta+\cos^2\theta)\sin\theta}{2Re(1-\cos\theta)}[E_1[Re(1-\cos\theta)]+\ln[Re(1-\cos\theta)]$$
$$+\gamma-Re(1-\cos\theta)]-\frac{(2+\cos\theta+\cos^2\theta)\sin\theta}{2Re(1+\cos\theta)}[E_1[Re(1+\cos\theta)]$$
$$+\ln[Re(1+\cos\theta)]+\gamma-Re(1+\cos\theta)]+\tfrac{1}{2}\sin 2\theta\Bigg[E_1[Re(1-\cos\theta)]$$
$$-\frac{e^{-Re(1-\cos\theta)}-1}{Re(1-\cos\theta)}+\ln(1-\cos\theta)+E_1[Re(1+\cos\theta)]$$
$$-\frac{e^{-Re(1+\cos\theta)}-1}{Re(1+\cos\theta)}+\ln(1+\cos\theta)+2(\gamma+\ln\tfrac{1}{4}Re)\Bigg]-\tfrac{3}{2}\sin 2\theta\Bigg\}\frac{1}{\sin 2\theta}. \tag{6.11}$$

From (6.6) the torque is found to be $(0,0,G)$ where

$$\frac{G}{\mu Ua^2} = -2\pi\left(\frac{1}{\ln\kappa}\right)^2\left[F_G(Re;\theta)+2\sin\theta\int_{-1}^{1}s\ln R_s(s)\,\mathrm{d}s\right]+O\left(\frac{1}{\ln\kappa}\right)^3, \tag{6.12}$$

where

$$F_G(Re;\theta) = \Bigg\{\frac{\cos\theta}{Re(1-\cos\theta)}\Bigg[2+2\frac{e^{-Re(1-\cos\theta)}-1}{Re(1-\cos\theta)}-E_1[Re(1-\cos\theta)]$$
$$-\ln[Re(1-\cos\theta)]-\gamma\Bigg]+\frac{\cos\theta}{Re(1+\cos\theta)}\Bigg[2+2\frac{e^{-Re(1+\cos\theta)}-1}{Re(1+\cos\theta)}$$
$$-E_1[Re(1+\cos\theta)]-\ln[Re(1+\cos\theta)]-\gamma\Bigg]$$
$$-2\Bigg[\frac{1}{Re(1-\cos\theta)}\left(1-\frac{1-e^{-Re(1-\cos\theta)}}{Re(1-\cos\theta)}\right)-\frac{1}{Re(1+\cos\theta)}$$
$$\times\left(1-\frac{1-e^{-Re(1+\cos\theta)}}{Re(1+\cos\theta)}\right)\Bigg]\Bigg\}\sin\theta. \tag{6.13}$$

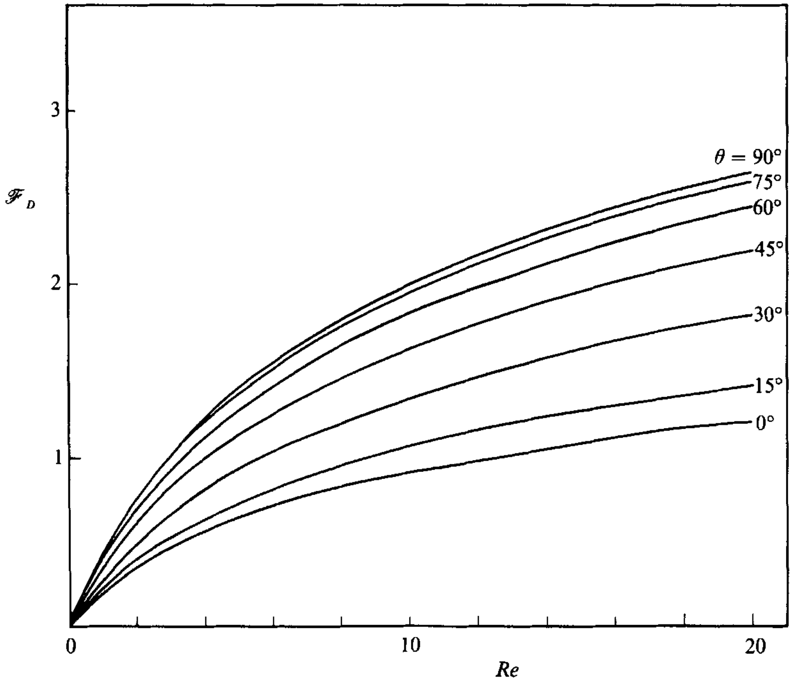

FIGURE 7. The behaviour of $\mathscr{F}_D$ as a function of Re for various values of θ.

The values of the integrals

$$\int_{-1}^{+1} \ln R_s(s)\,\mathrm{d}s \quad \text{and} \quad \int_{-1}^{+1} s \ln R_s(s)\,\mathrm{d}s$$

appearing in (6.8), (6.10) and (6.12) may be readily calculated for a given body shape, with the latter integral being zero for bodies with fore–aft symmetry.

It is convenient to recast the expressions for the drag and lift, namely (6.8) and (6.10), into the following forms:

$$D = D_s\left[1 + \frac{\mathscr{F}_D(Re;\theta)}{\ln \kappa^{-1}}\right] + O\left(\frac{1}{\ln \kappa}\right)^3 \tag{6.14}$$

for the drag, and for the lift

$$L = L_s\left[1 + \frac{\mathscr{F}_L(Re;\theta)}{\ln \kappa^{-1}}\right] + O\left(\frac{1}{\ln \kappa}\right)^3, \tag{6.15}$$

where D_s and L_s are the Stokes' drag and lift given by

$$\frac{D_s}{\mu U a} = \frac{-4\pi(2-\cos^2\theta)}{\ln \kappa + F_D(Re=0;\theta) + \frac{1}{2}\int_{-1}^{1} \ln R_s(s)\,\mathrm{d}s} + O\left(\frac{1}{\ln \kappa}\right)^3 \tag{6.16}$$

and

$$\frac{L_s}{\mu U a} = \frac{2\pi \sin 2\theta}{\ln \kappa + F_L(Re=0;\theta) + \frac{1}{2}\int_{-1}^{1} \ln R_s(s)\,\mathrm{d}s} + O\left(\frac{1}{\ln \kappa}\right)^3. \tag{6.17}$$

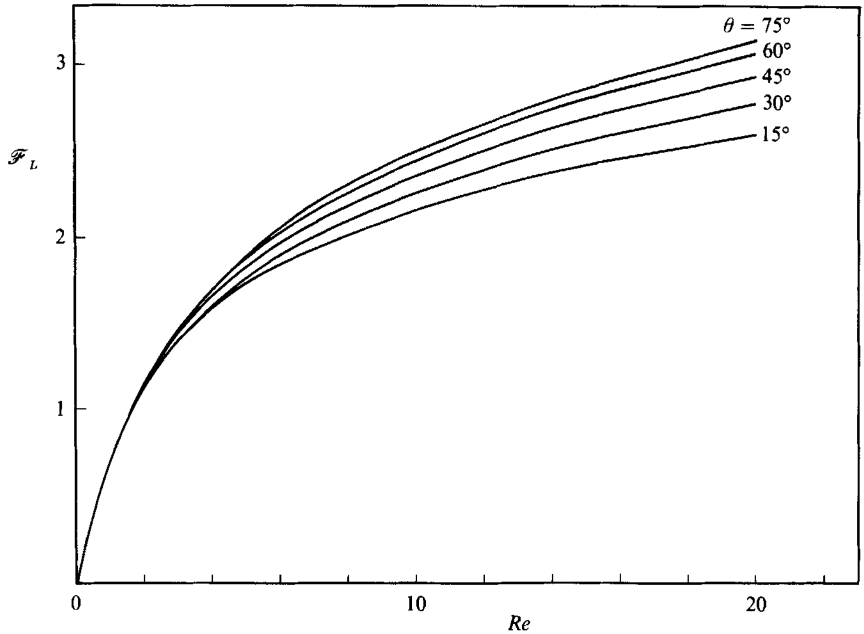

FIGURE 8. Behaviour of $\mathscr{F}_L$ as a function of Re for various values of θ.

The functions $\mathscr{F}_D$ and $\mathscr{F}_L$ are defined by

$$\mathscr{F}_D(Re;\theta) = F_D(Re;\theta) - F_D(0;\theta), \tag{6.18a}$$

and

$$\mathscr{F}_L(Re;\theta) = F_L(Re;\theta) - F_L(0;\theta). \tag{6.18b}$$

The behaviour of $\mathscr{F}_D$, $\mathscr{F}_L$ and F_G as functions of Re, for different orientations ($\theta = 0°$, 15°, 45°, 75°, 90°) is plotted in figures 7, 8 and 9 respectively. From figure 7 it is readily seen that $\mathscr{F}_D$ is a monotonically increasing function of Re for any orientation, indicating, as expected, an increase in the magnitude of the drag D as Re increases for a fixed value of κ (see (6.14)). Note also (figure 8) that the lift decreases (in magnitude) with increasing values of Re (see (6.15)).

As far as the torque on the body is concerned, it is seen from (6.12) that the effect of fluid inertia is to cause an additional torque on the body of

$$\frac{G}{\mu U a^2} = -2\pi \left(\frac{1}{\ln \kappa}\right)^2 F_G(Re;\theta) \tag{6.19}$$

with the remaining term in (6.12) representing the torque on the body which would exist at $Re = 0$ (for bodies lacking fore–aft symmetry). For bodies with fore–aft symmetry the torque is just that given by (6.19) so that its behaviour can be determined from $F_G(Re;\theta)$ plotted in figure 9. Thus for such bodies it is observed that, for any orientation, as the Reynolds number is increased from zero, the torque increases (in magnitude) from zero to reach a maximum at a relatively low Reynolds number (in the range 3 to 10), and then to decrease and vanish asymptotically as $Re \to \infty$ (not shown). For a body possessing fore–aft symmetry and sedimenting, there seem to be only two orientations for which the torque G is zero: $\theta = 0°$ and $\theta = 90°$,

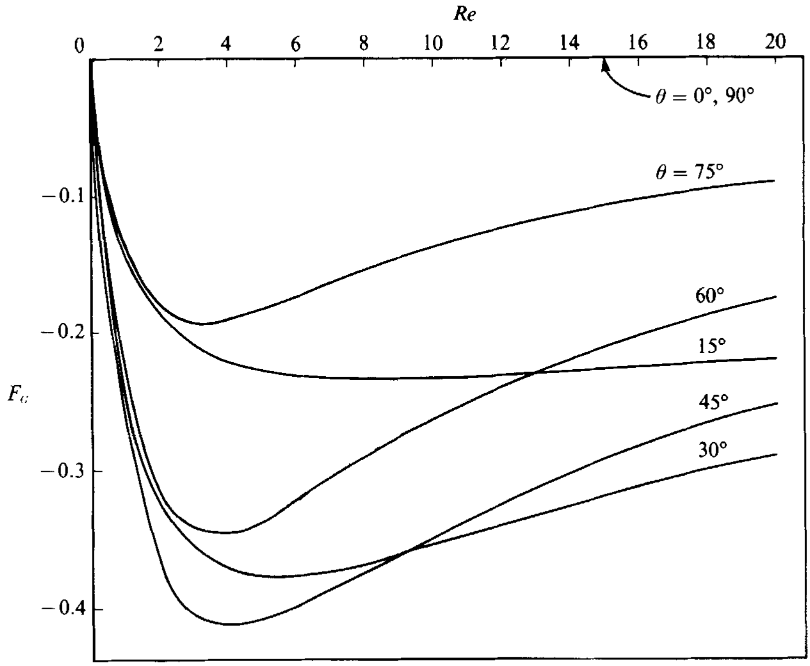

FIGURE 9. Behaviour of F_G as a function of Re for various values of θ.

i.e. the body centreline is respectively in the vertical or horizontal direction. The search for other orientations for which G vanishes (other than $Re = 0$) is difficult to carry out analytically; however, F_G was calculated numerically for every 5° in the range from 0° to 90° as well as being shown to be negative from the derived asymptotic forms of F_G from $\theta \rightarrow 0°$ and for $\theta \rightarrow 90°$ (not shown). Thus for $0 < \theta < \frac{1}{2}\pi$, the torque G is positive and for $\frac{1}{2}\pi < \theta < \pi$, G is negative, so that as the body sediments, it will rotate to the equilibrium horizontal orientation $\theta = \frac{1}{2}\pi$. This orientation is stable whilst the equilibrium vertical orientation $\theta = 0$ is unstable. This result is similar to that obtained by Cox (1965) for a spheroid of small eccentricity (see also Leal 1980). For the case of the body not possessing fore–aft symmetry the situation is different since the integral in (6.12) does not vanish and also the centre of gravity of the body is not at its midpoint. Furthermore, for such a sedimenting body the torque due to gravity (and buoyancy) about the origin ($s = 0$) is of order $1/\ln \kappa$ (since the drag force which balances the gravity force is of order $1/\ln \kappa$) and hence dominates over the hydrodynamic torque given by (6.12). Thus it follows that such bodies sediment with their centreline in the vertical direction and their centre of gravity below their midpoint $s = 0$. Intermediate situations for bodies deviating slightly from fore–aft symmetry can exist and may be expected to possess equilibrium orientations at values of θ different from $\theta = 0$ and $\frac{1}{2}\pi$. These situations require further investigation, and will not be considered here.

6.2. *The low-Reynolds-number limit*

The expressions for D, L and G for the body of circular cross-section are now considered for the case where Re is small. More specifically we seek the values of the functions $F_D(Re;\theta)$, $F_L(Re;\theta)$ and $F_G(Re;\theta)$ given by (6.9), (6.11) and (6.13) respectively in the limit $Re \to 0$. These are

$$F_D(Re;\theta) \sim \{2(1-4\ln 2)+(1+4\ln 2)\cos^2\theta + [2-\tfrac{1}{4}(\cos^2\theta+5)\cos^2\theta]Re\}\frac{1}{2(2-\cos^2\theta)}+O(Re^2), \quad (6.20)$$

$$F_L(Re;\theta) \sim -\tfrac{1}{2}-2\ln 2+(7+\cos^2\theta)\tfrac{1}{8}Re+O(Re^2), \quad (6.21)$$

and

$$F_G(Re;\theta) \sim -\tfrac{5}{12}(\sin 2\theta)\,Re+O(Re^2). \quad (6.22)$$

Brenner & Cox (1963) obtained an expression for the force exerted on a body of arbitrary shape by a uniformly moving fluid at small Reynolds number. This force may be written in dimensionless form as

$$F_i = 6\pi[\delta_{ij}+\tfrac{3}{16}Re\{3\phi_{ij}-\delta_{ij}(\phi_{kl}e_k e_l)\}]\,\phi_{jm}e_m, \quad (6.23)$$

where ϕ_{ij} is the Stokes resistance tensor for the body and $\boldsymbol{e}_i$ the unit vector in the flow direction. Substituting into (6.23) the value of ϕ_{ij} for a slender body obtained by Cox (1970) expressions for F_D and F_L may be obtained which are identical to those given by (6.20) and (6.21).

7. The case of an infinite cylinder

The special case of an infinite cylinder of arbitrary cross-sectional shape is considered in the present section. The force per unit length exerted by the fluid on the body is calculated; in particular, the expression for the resistance of a straight cylinder with uniform circular cross-section is compared to that obtained by Proudman & Pearson (1957) valid for small Reynolds number R based on the radius of the cylinder.

In terms of the present theory the force per unit length on an infinite cylinder is obtained by taking the limit $\kappa \to 0$ keeping $R = |U|b/\nu$ fixed but small. Upon replacing Re in (6.2) by R/κ and taking the limit as $\kappa \to 0$, the force per unit length becomes

$$\frac{\boldsymbol{f}(s)}{2\pi\mu a} = \frac{\boldsymbol{C}}{\ln\kappa}+\frac{\boldsymbol{D}-\boldsymbol{C}(\ln R-\ln\kappa)}{(\ln\kappa)^2}+O\left(\frac{1}{\ln\kappa}\right)^3, \quad (7.1)$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are analytic at $\kappa = 0$. For the present expansion procedure to be valid the second term in the expansion must be much smaller than the first as $\kappa \to 0$, i.e.

$$\left|\frac{\ln R-\ln\kappa}{(\ln\kappa)^2}\right| \ll \left|\frac{1}{\ln\kappa}\right| \quad \text{as} \quad \kappa \to 0. \quad (7.2)$$

This shows that for the above theory to be valid, R must satisfy

$$R < \kappa^{1-\delta} \quad \text{as} \quad \kappa \to 0, \quad (7.3)$$

where δ is a fixed positive constant much smaller than unity. Therefore for the case of the infinite cylinder considered by Proudman & Pearson for which $\kappa = 0$ with R fixed and small, the condition (7.3) is violated. It is obvious that under these conditions the result of Proudman & Pearson cannot be obtained. Thus some modifications to the present theory are required to obtain an expression for the resistance force on a slender body in the limit $\kappa \to 0$ with R fixed and small (i.e. when

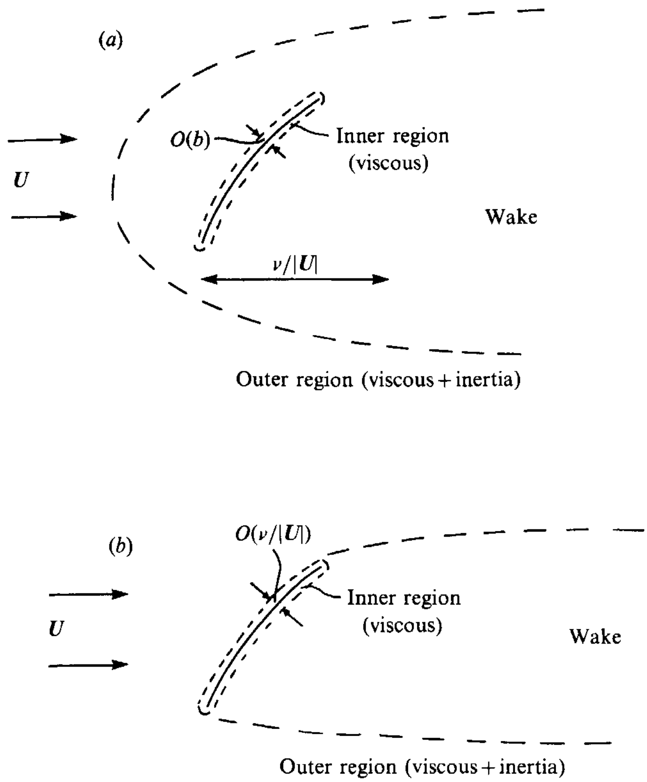

FIGURE 10. Inner and outer regions of expansions for (a) expansion in $(\ln \kappa)^{-1}$ and (b) expansion in $(\ln R)^{-1}$.

(7.3) is violated). It turns out that these modifications are minor and the procedure necessary is described below.

We first consider the case of a cylinder of finite length (with centreline not necessarily straight and cross-section not necessarily circular) and seek an expansion for the force per unit length in terms of $(\ln R)^{-1}$ rather than $(\ln \kappa)^{-1}$ in the limit as $R \rightarrow 0$. In the outer region the characteristic length is now taken to be $\nu/|U|$ (whereas previously it was taken to be a). A comparison between the regions of expansion for the expansion in $(\ln \kappa)^{-1}$ and for the expansion in $(\ln R)^{-1}$ is shown diagrammatically in figure 10. We denote by $\tilde{s} = s'|U|/\nu$ the new dimensionless arclength, so that $\tilde{s} = sRe$. The tilde over any variable indicates its value in this new outer region (i.e. made dimensionless with respect to $\nu/|U|$) so that $\tilde{\boldsymbol{u}} = \boldsymbol{u}$ and $\tilde{p} = p/Re$. In the inner region the characteristic length remains b. Consequently the inner governing equations together with the boundary conditions are exactly as considered previously in §3; inertia effects are neglected since they are of $O(R)$. The inner boundary conditions for the outer solution remain exactly as in (3.6) with $\ln \kappa$ replaced by $\ln R$. In the outer region the governing equations become

$$\tilde{\boldsymbol{u}} \cdot \tilde{\nabla} \tilde{\boldsymbol{u}} = -\tilde{\nabla} \tilde{p} + \tilde{\nabla}^2 \tilde{\boldsymbol{u}}; \quad \tilde{\nabla} \cdot \tilde{\boldsymbol{u}} = 0 \tag{7.4}$$

with the boundary condition

$$\tilde{\boldsymbol{u}} \rightarrow \boldsymbol{e} \quad \text{as} \quad \tilde{\boldsymbol{r}} \rightarrow \infty. \tag{7.5}$$

Expanding $\tilde{\boldsymbol{u}}$ and $\tilde{p}$ as

$$\tilde{\boldsymbol{u}} = \boldsymbol{e}+\frac{\tilde{\boldsymbol{u}}_1}{\ln R}+O\left(\frac{1}{\ln R}\right)^2, \quad \tilde{p} = \frac{\tilde{p}_1}{\ln R}+O\left(\frac{1}{\ln R}\right)^2 \tag{7.6}$$

and following exactly the same procedure as in §3.2 the force per unit length then becomes

$$\frac{\boldsymbol{f}(\tilde{s})}{2\pi\mu a} = \boldsymbol{e}\cdot(\boldsymbol{tt}-2\boldsymbol{I})\left(\frac{1}{\ln R}\right)+\left[\left(\tilde{\boldsymbol{J}}+\boldsymbol{e}\ln\frac{2\epsilon}{R_{\tilde{s}}}\right)\cdot(\boldsymbol{tt}-2\boldsymbol{I})\right.$$
$$\left.+\boldsymbol{e}\cdot(\tfrac{3}{2}\boldsymbol{tt}-\boldsymbol{I})+2\boldsymbol{e}\cdot\boldsymbol{K}(\tilde{s})+\boldsymbol{e}\cdot\boldsymbol{tt}\ln K(\tilde{s})\right]\left(\frac{1}{\ln R}\right)^2+O\left(\frac{1}{\ln R}\right)^3 \tag{7.7}$$

where $R_{\tilde{s}}$ is identical to R_s but expressed as a function of $\tilde{s}$ instead of s, and where

$$\tilde{J}_i = \left(\int_{-Re}^{\tilde{s}-\epsilon}+\int_{\tilde{s}+\epsilon}^{Re}\right)\mathscr{H}(Re = 1;\tilde{\boldsymbol{R}}-\hat{\tilde{\boldsymbol{R}}})\,\mathrm{d}\hat{\tilde{s}}. \tag{7.8}$$

The integrand $\mathscr{H}$ in this expression has exactly the same form as the one in (5.4) with Re and $\boldsymbol{R}-\hat{\boldsymbol{R}}$ substituted by 1 and $\tilde{\boldsymbol{R}}-\hat{\tilde{\boldsymbol{R}}}$ respectively. Upon changing variables in (7.8), one obtains

$$\tilde{J}_i = \left(\int_{-1}^{s-\epsilon^*}+\int_{s+\epsilon^*}^{1}\right)\mathscr{H}(Re;\boldsymbol{R}-\hat{\boldsymbol{R}})\,\mathrm{d}\hat{s}, \tag{7.9}$$

with $\epsilon^* = \epsilon/Re$. $\tilde{J}_i$ has exactly the same value as J_i but with ϵ^* instead of ϵ. We now show that there is an overlap of the domains of validity for the expressions (7.7) (for $R\rightarrow 0$ with $Re \neq 0$, fixed and finite) and (5.2) (for $\kappa\rightarrow 0$ with $Re \neq 0$, fixed and finite). The force $\boldsymbol{f}(s)$, given by (5.2), may be written in the form

$$\frac{\boldsymbol{f}(s)}{2\pi\mu a} = \frac{\boldsymbol{A}(s)}{\ln\kappa}+\frac{\boldsymbol{B}(s,Re)}{(\ln\kappa)^2}+O\left(\frac{1}{\ln\kappa}\right)^3, \tag{7.10}$$

where

$$\boldsymbol{A}(s) \equiv \boldsymbol{e}\cdot(\boldsymbol{tt}-2\boldsymbol{I}) \tag{7.11}$$

and

$$\boldsymbol{B}(s,Re) \equiv \left(\boldsymbol{J}+\boldsymbol{e}\ln\frac{2\epsilon}{R_s}\right)\cdot(\boldsymbol{tt}-2\boldsymbol{I})+\boldsymbol{e}\cdot(\tfrac{3}{2}\boldsymbol{tt}-\boldsymbol{I})+2\boldsymbol{e}\cdot\boldsymbol{K}+\boldsymbol{e}\cdot\boldsymbol{tt}\ln K. \tag{7.12}$$

From the expression for $\boldsymbol{J}$ given by (5.4) it may readily be shown that $\boldsymbol{B}(s,Re)$ tends to a finite limit as $Re\rightarrow 0$ (corresponding to the solution for $Re = 0$) and also that

$$\boldsymbol{B}(s,Re) \sim -(\ln Re)\,\boldsymbol{A}(s)+O(Re^0) \tag{7.13}$$

as $Re\rightarrow\infty$ (as may be shown by changing variables $s' = sRe$ and $\boldsymbol{R}' = Re\,\boldsymbol{R}$ in (5.4)).

Equation (7.7) can be written as

$$\frac{\boldsymbol{f}(s)}{2\pi\mu a} = \frac{\boldsymbol{A}(s)}{\ln R}+\frac{\boldsymbol{B}(s,Re)+\ln Re\,\boldsymbol{A}(s)}{(\ln R)^2}+O\left(\frac{1}{\ln R}\right)^3 \tag{7.14}$$

since as $\epsilon\rightarrow 0$, $\boldsymbol{J}(\epsilon) \sim -\boldsymbol{e}\ln\epsilon+\boldsymbol{A}_0+o(1)$, where $\boldsymbol{A}_0$ is a constant independent of ϵ, so that

$$\tilde{\boldsymbol{J}}(\epsilon) = \boldsymbol{J}\left(\frac{\epsilon}{Re}\right) = -\boldsymbol{e}\ln\left(\frac{\epsilon}{Re}\right)+\boldsymbol{A} = \boldsymbol{J}(\epsilon)+\boldsymbol{e}\ln Re. \tag{7.15}$$

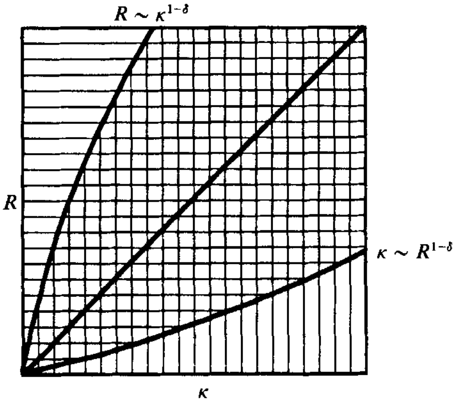

FIGURE 11. ▥ Domain of validity in (R, κ)-diagram of (7.10). ▤ Domain of validity in (R, κ)-diagram of (7.14). ▦ Common domain of validity of (7.10) and (7.14).

Expansion (7.10) is valid for $Re \rightarrow 0$, while expansion (7.14) is valid for $Re \rightarrow \infty$, since by (7.13), $\boldsymbol{B}(s, Re) + (\ln Re)\, \boldsymbol{A}(s)$ is finite in this latter limit. Now for Re fixed and as $\kappa \rightarrow 0$ one has

$$\frac{1}{\ln R} \sim \frac{1}{\ln \kappa} - \frac{\ln Re}{(\ln \kappa)^2} + O\left(\frac{1}{\ln \kappa}\right)^3 \tag{7.16a}$$

and

$$\frac{1}{(\ln R)^2} \sim \frac{1}{(\ln \kappa)^2} + O\left(\frac{1}{\ln \kappa}\right)^3, \tag{7.16b}$$

which upon substitution into (7.14) yields (7.10), showing that both expansions are valid for non-zero and fixed Re as $\kappa \rightarrow 0$. Thus there is a common domain of overlap between the two expansions. While (7.10) is valid only as long as (7.3) is satisfied, it may be shown in a similar manner that (7.14) is valid only as long as Re is not too small with

$$\kappa < R^{1-\delta}$$

(where δ is a fixed positive constant much smaller than unity). This condition will ensure that as $\kappa \rightarrow 0$, $R \rightarrow 0$, the second term in (7.14) is much smaller than the first. The regions of validity for (7.10) and (7.14) in the (R, κ)-plane are shown in figure 11. The force per unit length $\boldsymbol{f}(s)$ given by (7.10) and (7.14) can, in their domains of validity, be written respectively as

$$\frac{f_i(s)}{2\pi\mu U} = \left\{ \begin{array}{ll} \dfrac{A_i(s)}{\ln \kappa - \dfrac{B_i(s, Re)}{A_i(s)}} + O\left(\dfrac{1}{\ln \kappa}\right)^3 & (7.17a) \\[2ex] \dfrac{A_i(s)}{\ln R - \ln Re - \dfrac{B_i(s, Re)}{A_i(s)}} + O\left(\dfrac{1}{\ln R}\right)^3 & (7.17b) \end{array} \right\} \quad i = (1, 2, 3)$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are given by (7.11) and (7.12). Thus a uniformly valid expression for $\boldsymbol{f}(s)$ for the double limit $R \rightarrow 0$ and $\kappa \rightarrow 0$ is given by

$$\frac{f_i(s)}{2\pi\mu U} = \frac{A_i(s)}{\ln \kappa - \dfrac{B_i(s, Re)}{A_i(s)}} + O\left\{\left(\frac{1}{\ln \kappa}\right)^3, \left(\frac{1}{\ln R}\right)^3\right\}. \tag{7.18}$$

It is to be emphasized again that the equivalence of the two forms of expansions, namely (7.17*a*) and (7.17*b*) ceases to hold in the case of total absence of inertia, i.e. $Re = 0$, or that of an infinite cylinder, i.e. as $Re \to \infty$. In particular, when $Re \to 0$, (7.17*a*) must be used, and when $Re \to \infty$ it is (7.17*b*) which becomes valid although (7.18) can be used for either case. Thus results for drag, lift and torque written in the forms (6.8), (6.10) and (6.12) may be considered as being universally valid.

The force per unit length $f(s)$ (determined by (6.2)) on a straight circular cylinder with slowly varying cross-sectional radius $R_s(s)$ when compared with (7.10) gives the values of $\boldsymbol{A}(s)$ and $\boldsymbol{B}(s, Re)$. Taking the limit as $Re \to \infty$ and substituting into (7.14) yields the force per unit length in this limit as

$$\frac{f(s)}{2\pi\mu U} = \left(\frac{1}{\ln R}\right)(\cos\theta\boldsymbol{\beta} - 2\boldsymbol{e}) - \left(\frac{1}{\ln R}\right)^2 \{\tfrac{1}{2}(\cos\theta\boldsymbol{\beta} - 2\boldsymbol{e})\,[\ln(\sin^2\theta) + 2(\gamma - \ln 4) + 2\ln R_s(s)] - \cos\theta\boldsymbol{\beta} + \boldsymbol{e}\} + O\left(\frac{1}{\ln R}\right)^3. \quad (7.19)$$

Setting $\theta = \frac{1}{2}\pi$ and $R_s(s) = 1$ into (7.19), one obtains the drag force on an infinite circular cylinder of constant radius placed perpendicularly to the flow as

$$\frac{f(s)}{2\pi\mu U} = \frac{-2\boldsymbol{e}}{\ln R + \gamma - \frac{1}{2} - 2\ln 2} + O\left(\frac{1}{\ln\kappa}\right)^3 \quad (7.20)$$

which agrees with the result obtained by Proudman & Pearson (1957) for this case.

It is worth noting that (7.19) with $R_s(s) = 1$ gives the drag force per unit length on an infinite cylinder making an angle θ with the flow direction. However, this result ceases to be valid in the limit $\theta \to 0°$, i.e. when the body centreline is aligned with the flow direction: the term $\ln(\sin^2\theta)$ becomes singular. This is to be anticipated since in this case the boundary-layer thickness grows continuously without limit as the distance from the upstream cylinder end increases.

8. Discussion

The drag force on a slender spheroid with its symmetry axis placed perpendicular to the flow direction has been obtained theoretically by Chwang & Wu (1976) under the same conditions as has been assumed here, namely $\kappa \ll 1$, $R \ll 1$ with Re of order unity. For this case with $\theta = 90°$ and $R_s = (1-s^2)^{\frac{1}{2}}$, the value of D given by (6.8) gives

$$\frac{D}{8\pi\mu Ua} = \frac{1}{(\ln\kappa^{-1}) - \{E_1(Re) + Re^{-1}(1 - \mathrm{e}^{-Re}) + \ln(\frac{1}{2}Re) + \gamma - \frac{3}{2}\}} \quad (8.1a)$$

whilst that given by Chwang & Wu is

$$\frac{D}{8\pi\mu Ua} = \frac{1}{(\ln\kappa^{-1}) - \{E_1(\frac{1}{2}Re) + \ln(\frac{1}{4}Re) + \gamma - \frac{1}{2}\}}. \quad (8.1b)$$

The reason for this discrepancy is that Chwang & Wu were unable to perform a proper matching procedure at all points along the body axis (because they had previously assumed incorrectly that the drag force per unit length would be independent of position along the body axis). Instead they performed the matching at only one point on the body axis, namely the centre, and in so doing obtained a

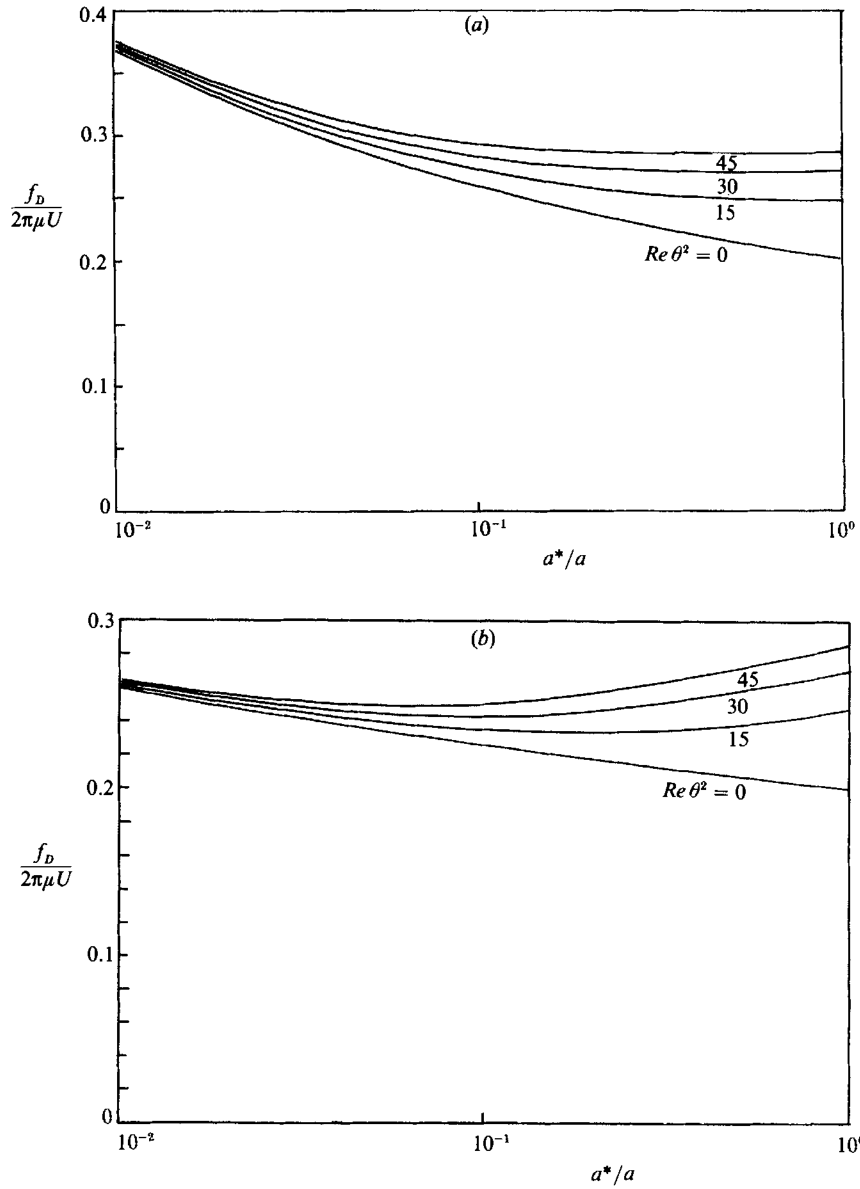

FIGURE 12 (a, b). For caption see facing page.

result (8.1 b) which was only approximately correct. For $Re \to 0$, (8.1 a) and (8.1 b) agree to order Re^{+1} with both giving

$$\frac{D}{8\pi\mu U a} \sim \frac{1}{(\ln \kappa^{-1}) + (\frac{1}{2} + \ln 2) - \frac{1}{2} Re + \dots}. \tag{8.2}$$

However for $Re \to \infty$, the present result (8.1 a) gives

$$\frac{D}{8\pi\mu U a} \sim \frac{1}{(\ln R^{-1}) - \gamma + \frac{3}{2} + \ln 2} \tag{8.3a}$$

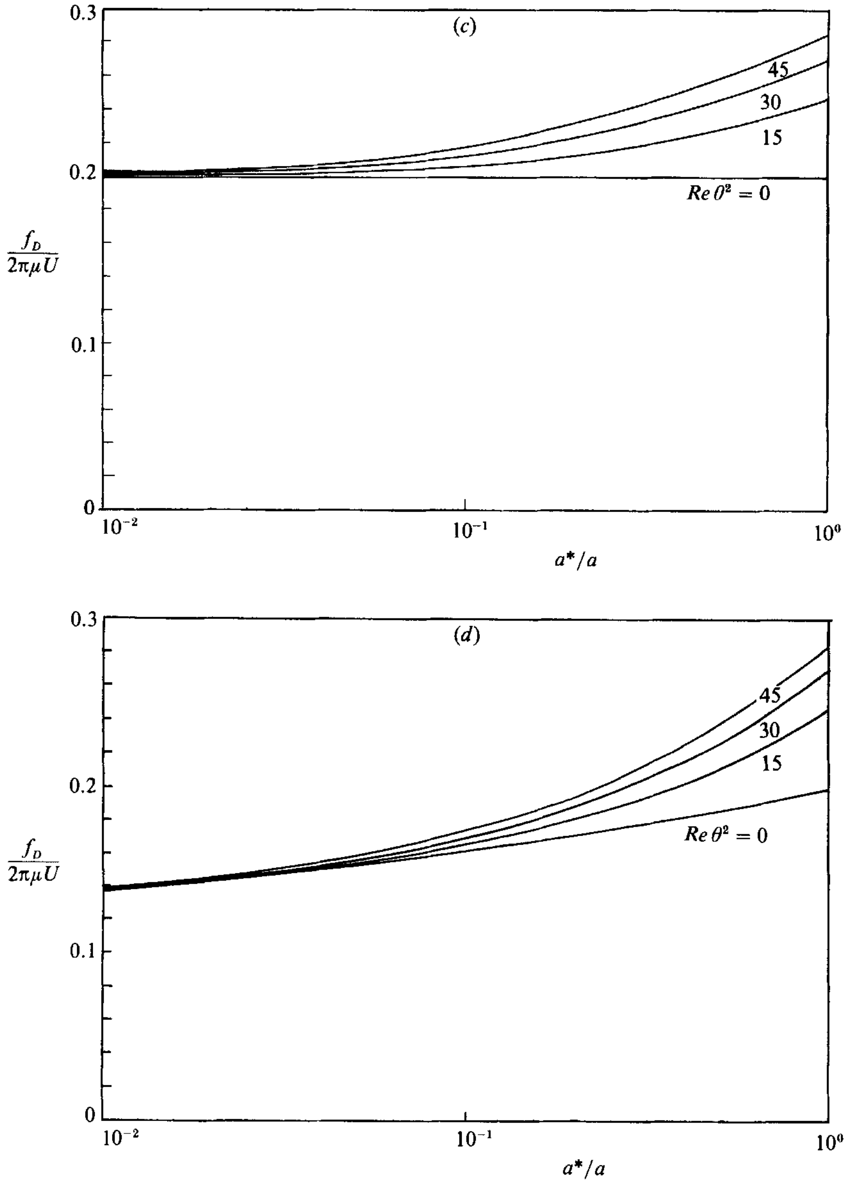

FIGURE 12. Dimensionless drag force per unit length $f_D/2\pi\mu U$ (given by (8.9)) plotted as a function of dimensionless distance a^*/a from the body nose for $R\kappa \equiv (b^2U/a\nu) = 10^{-4}$ for various body shapes: (*a*) $n = 0$, a cylinder; (*b*) $n = \frac{1}{4}$; (*c*) $n = \frac{1}{2}$, a paraboloid and (*d*) $n = 1$, a cone. Graphs are drawn for various values of $Re\,\theta^2$ (θ is in radians).

whereas that of Chwang & Wu, (8.1*b*), gives

$$\frac{D}{8\pi\mu Ua} \sim \frac{1}{(\ln R^{-1}) - \gamma + \frac{1}{2} + 2\ln 2}. \tag{8.3b}$$

In this limit $Re \to \infty$, if one takes the drag force per unit length of body as that given by (7.20) (obtained by Proudman & Pearson) with Reynolds number based on the local cross-sectional radius (i.e. R is replaced by $R\sqrt{1-s^2}$ in (7.20)) and integrates along the body axis, the drag force D obtained is that given by the present result (8.3a) (and not by (8.3b) obtained by Chwang & Wu).

An interesting use of the result (6.2) for the force per unit length on a slender body with straight centreline is the calculation of the way in which the drag force/unit length $f_D = \boldsymbol{f}\cdot\boldsymbol{e}$ varies along a slender body close to the nose $s = -1$ for when the body is aligned ($\theta = 0$) or almost aligned ($\theta \ll 1$) with the flow direction. We therefore consider

$$\theta \ll 1, \quad t \ll 1,$$

where $t = s+1$ is dimensionless distance back from the body nose. If the dimensional cross-sectional radius is $b^* = R_s \kappa a$ at a distance $a^* = ta$ from the body nose ($b^* \ll a^*$), then we obtain for $Re \to 0$

$$\frac{f_D}{2\pi\mu U} \sim \left\{\ln\frac{(aa^*)^{\frac{1}{2}}}{b^*} + \tfrac{3}{2}\ln 2 - \tfrac{1}{2}\right\}^{-1} + O(\theta^2, Re). \tag{8.4}$$

This, as expected, depends on the body length (however long it may be) and does not change rapidly with θ. However for $Re \to \infty$,

$$\frac{f_D}{2\pi\mu U} \sim \left[\ln\left(\frac{\nu}{U^{\frac{1}{2}}}\frac{a^{*\frac{1}{2}}}{b^*}\right) + \ln 2 - \gamma - \frac{1}{2}\left\{E_1\left(\frac{a^*U}{4\nu}\theta^2\right) + \ln\left(\frac{a^*U}{4\nu}\theta^2\right)\right\}\right]^{-1}, \tag{8.5}$$

which is independent of a and changes rapidly when θ is of order $(a^*U/\nu)^{-\frac{1}{2}}$ (which is small if $a^*U/\nu \gg 1$). This is expected since now a boundary layer forms at the body nose so that the flow behind the nose is not affected by the flow downstream (and hence by the length of the body). In addition, changes in flow occur for $\theta \approx (a^*U/\nu)^{-\frac{1}{2}}$ since this is the value of θ for which, at a distance a^* behind the nose, the body passes outside of its own wake. It is to be noted that the result (8.5) takes the form

$$\frac{f_D}{2\pi\mu U} \sim \left[\ln\left(\frac{\nu}{U^{\frac{1}{2}}}\frac{a^{*\frac{1}{2}}}{b^*}\right) + \ln 2 - \tfrac{1}{2}\gamma\right]^{-1} \quad \text{for} \quad \theta \ll \left(\frac{a^*U}{\nu}\right)^{-\frac{1}{2}}$$

and the form

$$\frac{f_D}{2\pi\mu U} \sim \left[\ln\left(\frac{b^*U}{\nu}\theta\right)^{-1} + 2\ln 2 - \gamma\right]^{-1} \quad \text{for} \quad \theta \gg \left(\frac{a^*U}{\nu}\right)^{-\frac{1}{2}}.$$

For body nose shapes of the form

$$b^*/b = (a^*/a)^n \quad \text{where } n \geqslant 0 \tag{8.8}$$

(8.5) reduces to

$$\frac{f_D}{2\pi\mu U} \sim \left[\ln (R\kappa)^{-\frac{1}{2}} - (n - \tfrac{1}{2})\ln\left(\frac{a^*}{a}\right) + \ln 2 - \gamma - \frac{1}{2}\left\{E_1\left(\tfrac{1}{4}Re\,\theta^2\left(\frac{a^*}{a}\right)\right) + \ln\left(\tfrac{1}{4}Re\,\theta^2\left(\frac{a^*}{a}\right)\right)\right\}\right]^{-1}, \tag{8.9}$$

where $R\kappa = (bU/\nu)(b/a) \ll 1$. This dimensionless drag force per unit length has been plotted as a function of the dimensionless distance (a^*/a) from the nose, for various body shapes ($n = 0, \frac{1}{4}, \frac{1}{2}, 1$) in figure 12, from which it is observed that the drag per unit length can either decrease ($n = 0$), increase ($n = \frac{1}{2}, 1$) or pass through a minimum ($n = \frac{1}{4}$) as one moves downstream from the nose. If f_L, the r_2 component of $\boldsymbol{f}$ (see figure 6) is the lift force per unit length, then calculations show that $(-f_L/2\pi\theta)$ has qualitatively the same behaviour as f_D.

The behaviour of the drag and lift forces per unit length on the body observed above for $Re \to \infty$ with θ small is reflected by the total drag D, lift L and torque G on the body. From (6.8)–(6.13) we can obtain for $\theta \to 0$ with a general value of Re

$$\frac{D}{4\pi\mu aU} \sim \left[\ln(\kappa^{-1}) - \frac{1}{2}\left\{\int_{-1}^{+1} \ln R_s \, ds + E_1(2Re) + \ln(2Re) + \gamma + 2 - 4\ln 2 + \frac{1}{2Re}(E_1(2Re) + \ln(2Re) + \gamma + 1 - 2Re - e^{-2Re})\right\}\right]^{-1}, \quad (8.10)$$

$$\frac{L}{4\pi\mu aU} \sim -\theta\left[\ln(\kappa^{-1}) - \frac{1}{2}\left\{\int_{-1}^{+1} \ln R_s \, ds + E_1(2Re) + \ln(2Re) + \gamma - 4\ln 2 - Re^{-1}(E_1(2Re) + \ln(2Re) + \gamma - \tfrac{1}{2} + \tfrac{1}{2}e^{-2Re})\right\}\right]^{-1}, \quad (8.11)$$

$$\frac{G}{2\pi\mu a^2 U} \sim \theta(\ln(\kappa^{-1}))^{-2}\left[-2\int_{-1}^{+1} s \ln R_s \, ds + 1 + \frac{1}{2Re}\left\{E_1(2Re) + \ln(2Re) + \gamma - 4 + 2Re^{-1}(1 - e^{-2Re})\right\}\right]. \quad (8.12)$$

These results (8.10)–(8.12) are not valid for $\theta \neq 0$ with $Re \gg 1$ (for which $Re\,\theta^2$ is of order unity), the equations (6.8)–(6.13) then yielding the following results (valid for the entire range of Re):

$$\frac{D}{4\pi\mu aU} \sim \Bigg[\ln(\kappa^{-1}) - \frac{1}{2}\Bigg\{\int_{-1}^{+1} \ln R_s \, ds + E_1(\tfrac{1}{2}Re\,\theta^2) + \ln(\tfrac{1}{2}Re\,\theta^2) + 2\gamma - 3\ln 2 - \frac{2}{Re\,\theta^2}(e^{-\frac{1}{2}Re\,\theta^2} - 1) + E_1(2Re) + \ln 2Re - \ln 2 - \frac{1}{2Re}(e^{-2Re} - 1) + \frac{1}{2Re}[E_1(\tfrac{1}{2}Re\,\theta^2) + \ln(\tfrac{1}{2}Re\,\theta^2) + 2\gamma - \tfrac{1}{2}Re\,\theta^2 + E_1(2Re) + \ln(2Re) - 2Re] + 1\Bigg\}\Bigg]^{-1}, \quad (8.13)$$

$$\frac{L}{4\pi\mu aU} \sim -\theta\Bigg[\ln(\kappa^{-1}) - \frac{1}{2}\Bigg\{\int_{-1}^{+1} \ln R_s \, ds + \frac{2}{Re\,\theta^2}(E_1(\tfrac{1}{2}Re\,\theta^2) + \ln(\tfrac{1}{2}Re\,\theta^2) + \gamma - \tfrac{1}{2}Re\,\theta^2 + 1 - e^{-\frac{1}{2}Re\,\theta^2}) + E_1(\tfrac{1}{2}Re\,\theta^2) + \ln(\tfrac{1}{2}Re\,\theta^2) + 2\gamma - 1 - 4\ln 2 - \frac{1}{Re}[E_1(2Re) + \ln(2Re) + \gamma + \tfrac{1}{2}(e^{-2Re} - 1)] + E_1(2Re) + \ln(2Re)\Bigg\}\Bigg]^{-1}, \quad (8.14)$$

$$\frac{G}{2\pi\mu a^2 U} \sim \theta(\ln(\kappa^{-1}))^{-2}\left[-2\int_{-1}^{+1} s \ln R_s \, ds + \frac{2}{Re\,\theta^2}\{E_1(\tfrac{1}{2}Re\,\theta^2) + \ln(\tfrac{1}{2}Re\,\theta^2) + \gamma\} + \frac{1}{2Re}\left\{E_1(2Re) + \ln(2Re) + \gamma - 4 + \frac{2}{Re}(1 - e^{-2Re})\right\}\right]. \quad (8.15)$$

These results, for a finite cylinder ($R_s = 1$) with $\kappa = 10^{-3}$ have been plotted in figures 13, 14 and 15. Thus it is seen that in the limit of $\theta \to 0$, the dimensionless drag and lift increase monotonically with Re (as was observed in figures 7 and 8 for larger

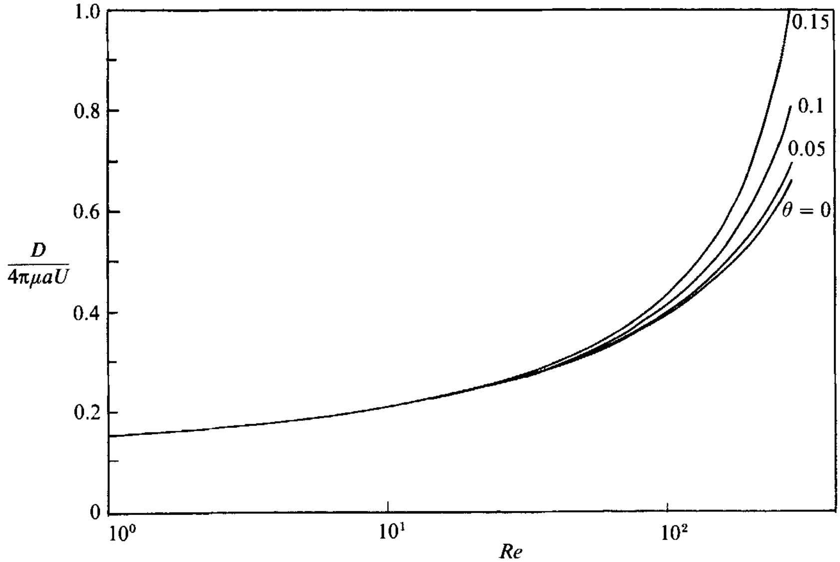

FIGURE 13. Dimensionless drag force ($D/4\pi\mu aU$) plotted as a function of Reynolds number $Re = aU/\nu$ for various values of θ for a circular cylinder [$R_s = 1$] with $\kappa = 10^{-3}$ (θ is in radians).

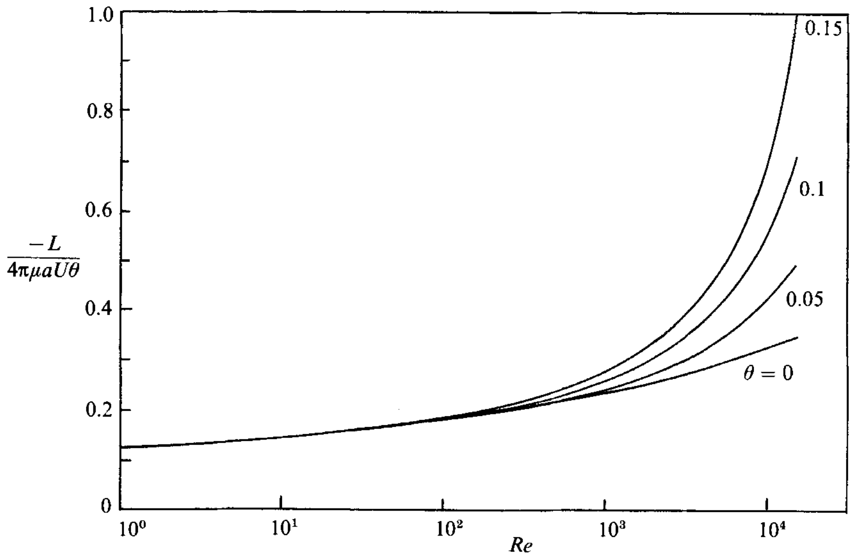

FIGURE 14. Dimensionless lift force ($-L/4\pi\mu aU\theta$) plotted as a function of Reynolds number $Re = aU/\nu$ for various values of θ for a circular cylinder with $\kappa = 10^{-3}$ (θ is in radians).

values of θ) but increase more rapidly with Re when θ is such that the body passes outside of its own wake ($Re\,\theta^2 \gg 1$). The dimensionless torque in the limit $\theta \to 0$ increases from zero at $Re = 0$ to a maximum value of $1.013(\ln(\kappa^{-1}))^2$ at $Re = 37.44$, thereafter tending asymptotically to a value of $(\ln(\kappa^{-1}))^2$ as $Re \to \infty$. However, when θ is such that the body passes outside of its own wake ($Re\,\theta^2 \gg 1$) the dimensionless torque instead tends to zero as $Re \to \infty$.

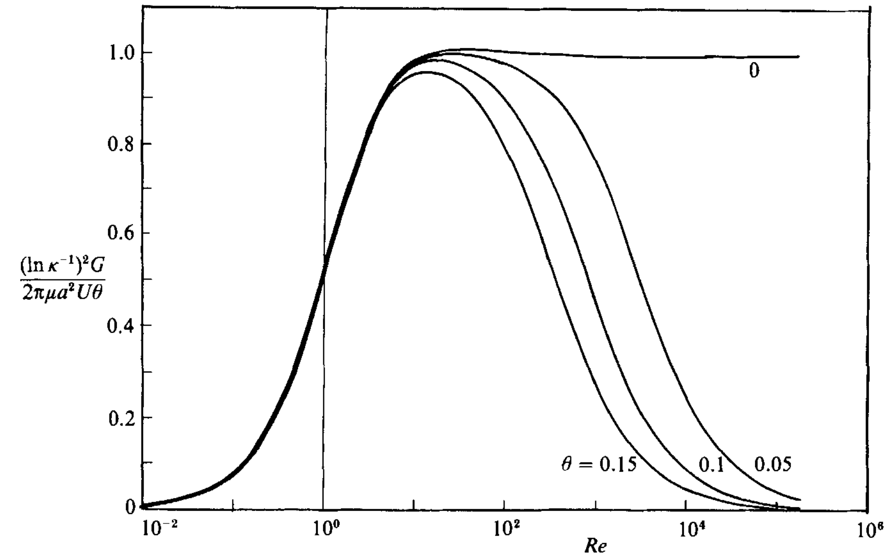

FIGURE 15. Dimensionless torque $(G(\ln \kappa^{-1})^2/2\pi\mu a^2 U\theta)$ plotted as a function of Reynolds number $Re = aU/\nu$ for various values of θ for a circular cylinder (θ is in radians).

If we consider the dimensionless torque $G/2\pi\mu a^2 U$ as a function of θ, then for $Re \to \infty$, equations (6.12) and (6.13) yield for a body with fore–aft symmetry

$$\frac{G}{2\pi\mu a^2 U} \sim (\ln(\kappa^{-1}))^2 Re^{-1} \cot\theta \times \{2\ln Re + 2\gamma + (1+\cos\theta)\ln(1-\cos\theta) + (1-\cos\theta)\ln(1+\cos\theta)\}. \quad (8.16)$$

For $Re \to \infty$, this possesses a maximum for a value of θ of

$$\theta \sim \surd 2\, e^{1-\frac{1}{2}\gamma} Re^{-\frac{1}{2}}, \quad (8.17)$$

for which

$$\frac{G}{2\pi\mu a^2 U} = (\ln(\kappa^{-1}))^2\, 2\surd 2\, e^{\frac{1}{2}\gamma - 1} Re^{-\frac{1}{2}}. \quad (8.18)$$

From this result (and from figure 9) and from (6.8) and (6.10), it is interesting to note that, considered as a function of θ, the drag D is a maximum for $\theta = \frac{1}{2}\pi$ and the lift $|L|$ a maximum for $\theta \approx \frac{1}{4}\pi$ for all values of the Reynolds number Re. However, the torque $|G|$ is a maximum at $\theta = \theta^*$, say, where θ^* decreases from $\frac{1}{4}\pi$ at $Re = 0$ to zero for $Re \to \infty$ (with asymptotic value given by (8.17)).

9. Concluding remarks

The general theory, presented in §§2–5, gives the force per unit length $f(s)$ acting on a long slender body of arbitrary cross-sectional shape and with curved centreline. The body is assumed placed in a uniform undisturbed flow. The ratio κ of cross-sectional dimension to body length and the Reynolds number R, based on the cross-sectional dimension, are assumed small. However, the results obtained, given by (5.2), were found to be valid for $\kappa \to 0$ and $R \to 0$ only if the inequality (7.3) is satisfied. In §7, these results were modified to obtain the force per unit length on the body (given by (7.18)) which was uniformly valid for all κ and R as $\kappa \to 0$, $R \to 0$. These results have been used to solve a variety of interesting problems. These include:

(i) The calculation of the force and torque acting upon a long slender rigid body placed in a uniform flow. This is carried out by direct application of (5.2). The case of a body with straight centreline and circular cross-section is presented in §6. In particular, the drag on the body is found to increase as the Reynolds number Re, based on the body half-length, increases for fixed κ, while the lift on the body is found to decrease as Re increases. It is also established that, for any value of Re, a body possessing fore–aft symmetry would, as it sediments, orientate itself so that its centreline is horizontal.

(ii) The flow at low Reynolds number (i.e. in the limit $Re \to 0$). The expressions for the drag and lift on the body for this situation are given in §6.2 and are found to agree with those obtained by Brenner & Cox (1963).

(iii) The case of flow around an infinite cylinder of arbitrary shape. This is considered in §7, where the force per unit length acting on the body is obtained from (7.17*b*) as an expansion in $1/\ln R$ instead of $1/\ln \kappa$, since for an infinite cylinder $\kappa = 0$. An expression for the force per unit length, given by (7.19) is obtained for an infinite cylinder at an arbitrary orientation relative to the flow. In particular, the force on an infinite circular cylinder placed in a cross-flow is determined and found to agree with the result obtained by Proudman & Pearson (1957).

(iv) The translation of a slender spheroid in a direction perpendicular to its symmetry axis. The present theory (§8) corrects earlier results obtained by Chwang & Wu (1976).

(v) The calculation of the force per unit length and also the total drag, lift and torque on a slender body (with straight centreline and circular cross-section) in a uniform flow in the limits of $Re \to 0$ and $Re \to \infty$ (§8). Of particular interest is the case of $Re \to \infty$ with the body centreline aligned or nearly aligned with the flow direction.

This work was supported by the Natural Sciences and Engineering Research Council of Canada under grant A7007.

REFERENCES

Batchelor, G. K. 1970 *J. Fluid Mech.* **44**, 419.

Brenner, H. & Cox, R. G. 1963 *J. Fluid Mech.* **17**, 561.

Chwang, A. T. & Wu, T. Y. 1976 *J. Fluid Mech.* **75**, 677.

Cox, R. G. 1965 *J. Fluid Mech.* **23**, 625.

Cox, R. G. 1970 *J. Fluid Mech.* **44**, 791.

Happel, J. & Brenner, H. 1963 *Low Reynolds Number Hydrodynamics.* Prentice-Hall.

Leal, L. G. 1980 *Ann. Rev. Fluid Mech.* **12**, 435.

Proudman, I. & Pearson, J. R. A. 1957 *J. Fluid Mech.* **2**, 237.